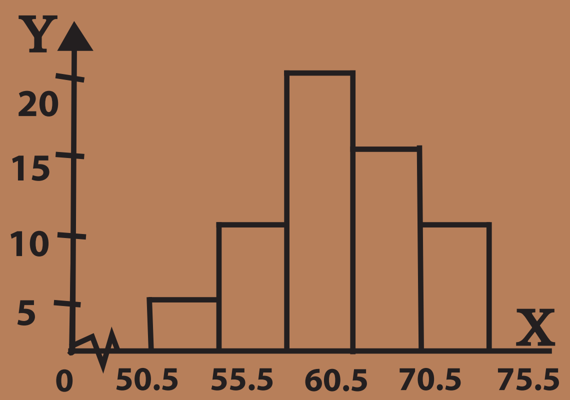
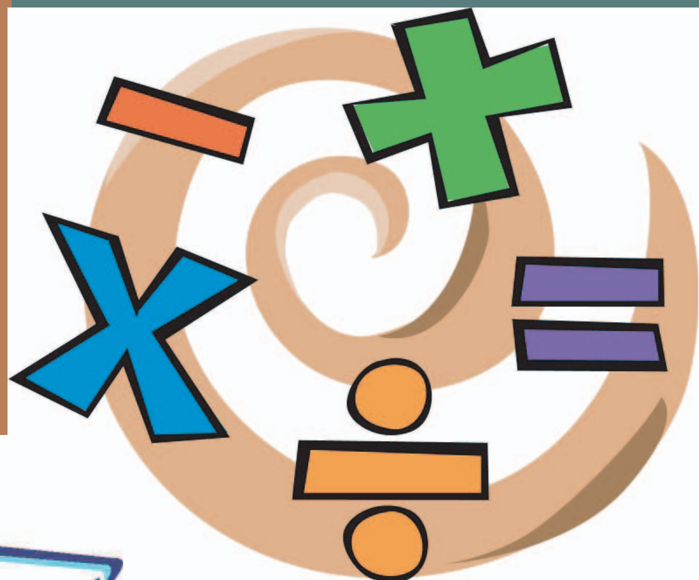
Mathematics

Classes 9-10





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Mathematics

Classes Nine-Ten

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Preface

Education is the pre-requisite for holistic development. In order to face the challenges of the fast changing world and to accelerate the development and prosperity of Bangladesh, there is a need for well-developed human resources. One of the most important objectives of Secondary Education is to develop students' intrinsic talents and potentials to build the country in line with the spirit of the Language Movement and the Liberation War. Besides, purpose of education at this stage is also to prepare students for higher levels of study by integrating and enhancing the basic knowledge and skills acquired at the primary level. The secondary level of education also takes into consideration the process of learning that helps students become skilled and worthy citizens in the backdrop of country's economic, social, cultural and environmental realities.

The new curriculum of secondary education has been developed keeping in mind the aims and objectives of the National Education Policy 2010. In the curriculum, national ideals, aims, objectives and demands of the time have been properly reflected. It will ensure also the learning of the students according to their age, talent and receptivity. In addition, a broad range starting from moral and human values of the students, awareness of history and culture, the Liberation War, arts-literature-heritage, nationalism, environment, religion-caste-creed and gender is given due importance. Everything is done in the curriculum to enable students to grow up a scientifically conscious nation to be able to apply science in every sphere of life and to realize the Vision of Digital Bangladesh 2021.

All textbooks are written in the light of this new curriculum. In the development of the textbooks, learners' ability, inclination aptitude and prior experience have been given due consideration. Special attention has been paid to the flourishing of creative talents of the students and for selecting and presenting the topics of the textbooks. In the beginning of every chapter, learning outcomes are added to indicate what they might learn. Various activities, creative questions and other tasks are included to make teaching-learning and assessment more creative and effective.

Mathematics plays an important role in developing scientific knowledge at this time of the 21st century. Not only that, the application of Mathematics has increased in family and social life including personal life. With all these things under consideration Mathematics has been presented easily and nicely at the lower secondary level to make it useful and delightful to the learners, and quite a number of mathematical topics have been included in the text book.

This textbook has been written keeping in mind the promise and vision of the 21st century and in accordance with the new curriculum. So, any constructive and logical suggestions for its improvement will be paid mentionable attention. Very little time was available for writing the textbook. As a result, there could be some unintentional mistakes in it. In the next edition of the book, more care will be taken to make the book more elegant and error free.

Thanks to those who have sincerely applied their talent and labour in writing, editing, translating, drawing, setting sample questions and publishing of the book. I hope the book will ensure happy reading and expected skill acquisition of the learners.

Professor Md. Mostafa Kamaluddin
Chairman
National Curriculum and Textbook Borard, Dhaka.

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Chapter One

Real Number

Mathematics is originated from the process of expressing quantities in symbols or numbers. The history of numbers is as ancient as the history of human civilization. Greek Philosopher Aristotle According to the formal inauguration of mathematics occurs in the practice of mathematics by the sect of priest in ancient Egypt. So, the number based mathematics is the creation of about two thousand years before the birth of Christ. After that, moving from many nations and civilization, numbers and principles of numbers have gained an universal form at present.

The mathematicians in India first introduce zero (0) and 10 based place value system for counting natural numbers, which is considered a milestone in describing numbers. Chinese and Indian mathematicians extended the idea zero, real numbers, negative number, integer and fractional numbers which the Arabian mathematicians accepted in the middle age. But the credit of expressing number through decimal fraction is awarded to the Muslim Mathematicians. Again they introduce first the irrational numbers in square root form as a solution of the quadratic equation in algebra in the 11th century. According to the historians, very near to 50 BC the Greek Philosophers also felt the necessity of irrational number for drawing geometric figures, especially for the square root of 2. In the 19th century European Mathematicians gave the real numbers a complete shape by systematization. For daily necessity, a student must have a vivid knowledge about 'Real Numbers'. In this chapter real numbers are discussed in detail.

At the end of this chapter, the students will be able to –

- Classify real numbers
- Express real numbers into decimal and determine approximate value
- Explain the classification of decimal fractions
- Explain recurring decimal numbers and express fractions into recurring decimal numbers
- Transform recurring decimal fraction into simple fractions
- Explain non-terminating non-recurring decimal fraction
- Explain non-similar and similar decimal fraction
- Add, subtract multiply and divide the recuring decimal fraction and solve various problems related to them.

Natural Number

$1, 2, 3, 4.........$ etc. numbers are called natural number or positive whole numbers. $2, 3, 5, 7.........$ etc. are prime numbers and $4, 6, 8, 9,.........$ etc. are composite numbers.

Integers

All numbers (both positive and negative) with zero (0) are called integers i.e. .... $-3, -2, -1, 0, 1, 2, 3.........$ etc. are integers.

Fractional Number

If p, q are co-prime numbers ; $q \neq 0$ and $q \neq 1$, numbers expressed in $\frac{p}{q}$ form are called fractional number.

Example : $\frac{1}{2}, \frac{3}{2}, \frac{-5}{3}$ etc. are fractional numbers.

If $p < q$, then it is a proper fraction and if $p > q$ then it is an improper fraction : Example $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, ...$ etc. proper and $\frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, ...$ etc. improper fraction.

Rational Number

If p and q are integers and $q \neq 0$, number expressed in the form $\frac{p}{q}$ is called rational number. For example : $\frac{3}{1} = 3, \frac{11}{2} = 5.5, \frac{5}{3} = 1.666...$ etc. are rational numbers. Rational numbers can be expressed as the ratio of two integers. So, all integers and all fractional numbers are rational numbers.

Irrational Number

Numbers which cannot be expressed in $\frac{p}{q}$ form, where p, q are integers and $q \neq 0$ are called Irrational Numbers. Square root of a number which is not perfect square, is an irrational number. For example: $\sqrt{2} = 1.414213.....$, $\sqrt{3} = 1.732.....$, $\frac{\sqrt{5}}{2} = 1.58113....$ etc. are irrational numbers. Irrational number cannot be expressed as the ratio of two integers.

Decimal Fractional Number

If rational and irrational numbers are expressed in decimal, they are known as decimal fractional numbers. As for instance, $3 = 3 \cdot 0, \frac{5}{2} = 2 \cdot 5, \frac{10}{3} = 3 \cdot 3333.......$, $\sqrt{3} = 1 \cdot 732.........$etc. are decimal fractional numbers. After the decimal, if the number of digits are finite, it is terminating decimals and if it is infinite it is known as non-terminating decimal number. For example, 0.52, 3.4152 etc. are terminating decimals and $1 \cdot 333.......$, $2 \cdot 123512367...........$ etc. are non-terminating decimals. Again, if the digits after the decimal of numbers are repeated among themselves, they are known recurring decimals and if they are not repeated, they are called non-recurring decimals. For example : $1\cdot 2323........$, $5\cdot\dot{6}\dot{5}\dot{4}$ etc. are the recurring decimals and $0\cdot 523050056........$, $2\cdot 12340314........$etc. are non-recurring decimals.

Real Number

All rational and irrational numbers are known as real numbers. For example :

$0, \pm 1, \pm 2, \pm 3, ...\quad \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{4}{3}, ...\quad \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}......$

$1\cdot 23,\ 0\cdot 415,\ 1\cdot 3333.......,\ 0\cdot\dot{6}\dot{2},\ 4\cdot 120345061..........$ etc. are real numbers.

Positive Number

All real numbers greater than zero are called positive numbers. As for instance

$1,\ 2,\ \frac{1}{2},\ \frac{3}{2},\ \sqrt{2},\ 0\cdot 415,\ 0\cdot\dot{6}\dot{2},\ 4\cdot 120345061.............$ etc. are positive numbers.

Negative Number

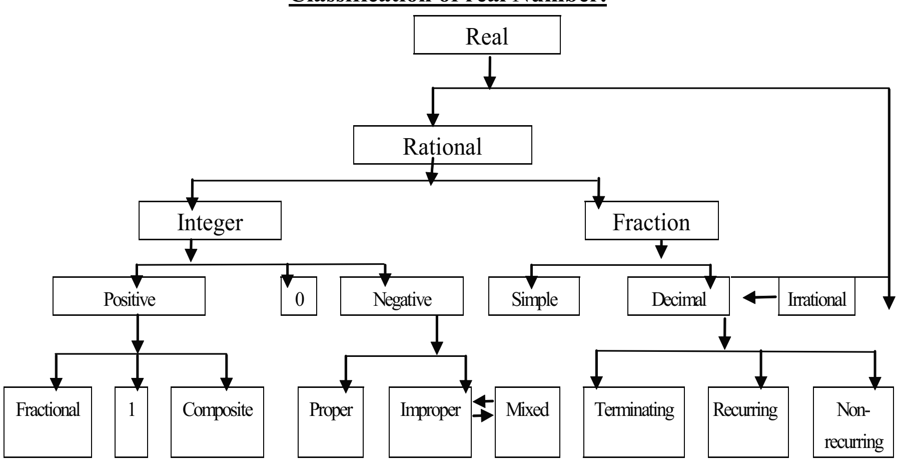
All real numbers less than zero are called negative numbers. For example, $-1,\ -2,\ -\frac{1}{2},\ -\frac{3}{2},\ -\sqrt{2},\ -0\cdot 415,\ -0\cdot\dot{6}\dot{2},\ -4\cdot 120345061.............$ etc. are negative numbers.

Non-Negative Number

All positive numbers including zero are called non-negative numbers. For example,

$0,\ 3,\ \frac{1}{2},\ \ 0\cdot 612,\ 1\cdot\dot{3},\ 2\cdot 120345.............$ etc. are non-negative numbers.

<u>Classification of real Number.</u>



Activity : Show the position of the numbers $\frac{3}{4}$, 5, -7, $\sqrt{13}$, 0, 1, $\frac{9}{7}$, 12, $2\frac{4}{5}$, $1\cdot1234......$, $.3\dot{2}\dot{3}$ in the classification of real numbers.

Example 1. Determine the two irrational numbers between $\sqrt{3}$ and 4.

Solution : Here, $\sqrt{3} = 1.7320508......$

Let, $a = 2.030033000333.....$

and $b = 2.505500555.......$

Clearly :both a and b are real numbers and both are greater than $\sqrt{3}$ and less than 4.

i.e., $\sqrt{3} < 2.03003300333......... < 4$

and $\sqrt{3} < 2.505500555............... < 4$

Again, a and b cannot be expressed into fractions.

$\therefore$ a and b are the two required irrational numbers.

Basic characteristics of addition and multiplication over a real number :

1. If a, b are real numbers, $(i)\ a + b$ is real and $(ii)\ ab$ is a real number
2. If a, b are real numbers, $(i)\ a + b = b + a$ and $(ii)\ ab = ba$
3. If a, b, c are real numbers, $(i)\ (a + b) + c = a + (b + c)$ and $(ii)\ (ab)c = a(bc)$
4. If a is a real number, in real numbers there exist only two number 0 and 1 where $(i)\ 0 \neq 1$ $(ii)\ a + 0 = a$ $(iii)\ a.1 = 1.a = a$
5. If a is a real number, $(i)\ a + (-a) = 0$ (ii) If $a \neq 0$, $a.\frac{1}{a} = 1$
6. If a, b, c are real numbers, $a(b + c) = ab + ac$
7. If a, b are real numbers, $a < b$ or $a = b$ or $a > b$
8. If a, b, c are real numbers and $a < b$, $a + c < b + c$

9 If a, b, c are real numbers and $a < b$, $(i)\ ac < bc$ where $c < 0$

(ii) If $ac > bc$, $c < 0$

Proposition : $\sqrt{2}$ is an irrational number.

We know,

$1 < 2 < 4$

$\therefore\ \sqrt{1} < \sqrt{2} < \sqrt{4}$

or, $1 < \sqrt{2} < 2$

Proof : $1^2 = 1$, $(\sqrt{2})^2 = 2$, $2^2 = 4$

$\therefore$ Therefore, the value of $\sqrt{2}$ is greater than 1 and less than 2.

$\therefore$ $\sqrt{2}$ is not an integer.

$\therefore$ $\sqrt{2}$ is either a rational number or a irrational number. If $\sqrt{2}$ is a rational number let, $\sqrt{2} = \frac{p}{q}$; where p and q are natural numbers and co-prime to each other and $q > 1$

or, $2 = \frac{p^2}{q^2}$; squaring

or, $2q = \frac{p^2}{q}$; multiplying both sides by q.

Clearly $2q$ is an integer but $\frac{p^2}{q}$ is not an integer because p and q are co-prime natural numbers and $q > 1$

$\therefore$ $2q$ and $\frac{p^2}{q}$ cannot be equal, i.e., $2q \neq \frac{p^2}{q}$

$\therefore$ Value of $\sqrt{2}$ cannot be equal to any number with the form $\frac{p}{q}$ i.e., $\sqrt{2} \neq \frac{p}{q}$

$\therefore$ $\sqrt{2}$ is an irrational number.

Example 2. Prove that, sum of adding of 1 with the product of four consecutive natural numbers becomes a perfect square number.

Solution : Let four consecutive natural numbers be $x,\ x+1,\ x+2,\ x+3$ respectively.
By adding 1 with their product we get,

$x(x+1)(x+2)(x+3)+1 = x(x+3)(x+1)(x+2)+1$
$= (x^2+3x)(x^2+3x+2)+1$

$= a(a+2)+1$; $[x^2+3x=a]$

$= a(a+2)+1$;

$= a^2+2a+1 = (a+1)^2 = (x^2+3x+1)^2$;

which is a perfect square number.

$\therefore$ If we add 1 with the product of four consecutive numbers, we get a perfect square number.

Activity : Proof that, $\sqrt{3}$ is an irrational number

Classification of Decimal Fractions

Each real number can be expressed in the form of a decimal fraction.

For example, $2 = 2\cdot 0,\ \frac{2}{5} = 0.4,\ \ \frac{1}{3} = 0.333....$ etc. There are three types of decimal fractions :terminating decimals, recurring decimals and non-terminating decimals.

Terminating decimals : In terminating decimals, the finite numbers of digits are in the right side of a decimal points. For example, $0.12, 1.023, 7.832, 54.67, ...$ etc. are terminating decimals.

Recurring decimals : In recurring decimals, the digits or the part of the digits in the right side of the decimal points will occar repeatedly. For example, $3.333....., 2.454545......, 5.12765765..........$ etc. are recurring decimals.

Non-terminating decimals : In non-terminating decimals, the digits in the right side of a decimal point never terminate, i.e., the number of digits in the right side of decimal point will not be finite neither will the part occur repeatedly. For example. $1.4142135......, 2.8284271.......$ etc. are non-terminating decimals.

Terminating decimals and recurring decimals are rational numbers and non-terminating decimals are irrational numbers. The value of an irrational number can be determined upto the required number after the decimal point. If the numerator and denominator of a fraction can be expressed in natural numbers, that fraction is a rational number.

> **Activity :** Classify the decimal fractions stating reasons :
> $1.723,\ 5.2333........,\ 0.0025,\ 2.1356124.......,\ 0.0105105........$ and $0.450123.......$

Recurring decimal fraction :

Expressing the fraction $\frac{23}{6}$ into decimal fractions, we get,

```
23
-- = 6 ) 23 ( 3·833
6         18
          --
           50
           48
           --
            20
            18
            --
             20
             18
             --
              20
              18
              --
               2
```

Observe : It is found that, the process of division is not ended at the time of dividing a numerator of a fraction by its denominator. To convert it into decimal fraction in the quotient 3'occurs repeatedly. Here, $3.8333.....$ is a recurring decimal fraction.

If digit or successive digit of a decimal fractions a in the right side of the decimal point appear again and again, these are called recurring decimal fractions.

If a digit or successive digits of decimal fractions.

In recurring decimal fractions, the portion which occurs again and again, is called recurring part. In recurring decimal fraction, if one digit recurs, the recurring point is used upon it and if more than one digits recurs, the recurring point is used only upon the first and the last digits.

As for example : $2.555.......$ is written as $2.\dot{5}$ and $3.124124124........$ is written as $3.\dot{1}2\dot{4}$.

In recurring decimal fractions, if there is no other digit except recurring one, after decimal point it is called pure recurring decimal and if there is one digit or more after decimal point in addition to recurring one, it is called mixed recurring decimal. For example, $1.\dot{3}$ is a pure recurring decimal and $4.235\dot{1}\dot{2}$ is a mixed recurring decimal.

If there exists prime factors other than 2 and 5 in the denominator of the fraction, the numerator will not be fully divisible by denominator. As the last digit of successive divisions cannot be other than $1, 2, ..., 9$, at one stage the same number will be repeated in the remainder. The number in the recurring part is always smaller than that of the denominator.

Example 3. Express $\frac{3}{11}$ into decimal fraction.

Solution :

11)30 (0.2727
22
80
77
30
22
80
77
3

Example 4. Express $\frac{95}{37}$ into decimal fraction.

Solution :

37)95 (2.56756
74
210
185
250
222
280
259
210
185
250
222
28

Required decimal fraction = $0.2727......$
$= 0.\dot{2}\dot{7}$

Required decimal fraction =
$2.56756...... = 2.\dot{5}6\dot{7}$

Conversion of Recurring Decimal into Simple Fraction

Determining the value of recurring fraction :

Example 5. Express $0.\dot{3}$ into simple fraction.

Solution : $0.\dot{3} = 0.3333........$

$0.\dot{3} \times 10 = 0.333....... \times 10 = 3.333......$

and $0.\dot{3} \times 1 = 0.333....... \times 1 = 0.333......$

subtracting, $0.\dot{3} \times 10 - 0.\dot{3} \times 1 = 3$

or, $0.\dot{3} \times (10 - 1) = 3$ or $0.\dot{3} \times 9 = 3$

Therefore, $0.\dot{3} = \frac{3}{9} = \frac{1}{3}$

Required fraction is $\frac{1}{3}$

Example 6. Express $0.\dot{2}\dot{4}$ into simple fraction.

Solution : $0.\dot{2}\dot{4} = 0 \cdot 24242424........$

So, $0.\dot{2}\dot{4} \times 100 = 0.242424....... \times 100 = 24.2424......$

and $0.\dot{2}\dot{4} \times 1 = 0.242424....... \times 1 = 0.242424......$

Subtracting $0.\dot{2}\dot{4}(100 - 1) = 24$

or, $0.\dot{2}\dot{4} \times 9 = 24$ or, $0.\dot{2}\dot{4} = \frac{24}{9} = \frac{8}{33}$

Required fraction is $\frac{8}{33}$

Example 7. Express $5.1\dot{3}4\dot{5}$ into simple fraction.

Solution : $5.1\dot{3}4\dot{5} = 5.1345345345.........$

So, $5.1\dot{3}4\dot{5} \times 10000 = 5.1345345...... \times 10000 = 51345.345......$

and $5.1\dot{3}4\dot{5} \times 10 = 5.1345345....... \times 10 = 51.345......$

Subtracting, $5.1\dot{3}4\dot{5} \times 9 = 51345 - 51$

So, $5.1\dot{3}4\dot{5} = \frac{51345 - 51}{9} = \frac{51294}{9} = \frac{8549}{1665} = 5\frac{224}{1665}$

Required fraction is $5\frac{224}{1665}$

Example 8. Express $42.34\dot{7}\dot{8}$ into simple fraction.

Solution : $42.34\dot{7}\dot{8} = 42.347878........$

So, $42.34\dot{7}\dot{8} \times 10000 = 42.347878.........\times 10000 = 42348.7878$

and $42.34\dot{7}\dot{8} \times 100 = 42.347878........ \times 100 = 4234.7878$

Subtracting, $42.34\dot{7}\dot{8} \times 9900 = 423478 - 4234$

Therefore, $42.34\dot{7}\dot{8} = \frac{423478 - 4234}{9900} = \frac{419244}{9900} = \frac{34937}{825} = 42\frac{287}{825}$

Required fraction is $42\frac{287}{825}$

Explanation : From the examples 5, 6, 7 and 8 , it appears that,

- The recurring decimal has been multiplied by the number formed by putting at the right side of 1 the number of zeros equal to the number of digits in the right side of decimal point in the recurring decimal.
- The recurring decimal has been multiplied by the number formed by putting at the right side of 1 the number of zeros equal to the number of digits which are non-recurring after decimal point of the recurring decimal.
- the second product has been subtracted from the first product. By subtracting the second product from the first product the whole number has been obtained at the right side. Here it is observed that, the number of non-recurring part has been subtracted from the number obtained by removing the decimal and recurring points of recurring decimal fraction.
- The result of subtraction has been divided by the number formed by writing the same number of 9s equal to the number of digits of recurring part at the left and number of zeros equal to the number of digits of non-recurring part at the right.
- In the recurring decimals, converting into fractions the denominator is the number of 9s equal to the number of digits in the rec urring part and in right side of all 9s number of zeros equal to the number of digits in the non-recurring part. And the numerator in the result that is obtained by subtracting the number of the digits formed by omitting the digits of recurring part from the number formed by removing the decimal and recurring points of recurring decimal.

Remark : Any recurring decimal can also be converted into a fraction. All recurring decimals are rational numbers.

Example 9. Express $5.23\dot{4}5\dot{7}$ into simple fraction.

Solution : $5.23\dot{4}5\dot{7} = 5.23457457457.........$

So, $5.23\dot{4}5\dot{7} \times 100000 = 523457.457457$

and $5.23\dot{4}5\dot{7} \times 100 = 523.457457$

Subtracting, $5.23\dot{4}5\dot{7} \times 99900 = 522934$

Therefore, $5.23\dot{4}5\dot{7} = \frac{522934}{99900} = \frac{261467}{49950}$

Required fraction is $\frac{261467}{49950}$

Explanation : Here in the decimal part the recurring decimal has been multiplied first by 100000 (5 zeros at the right side of 1) as there are two digits at the left side of recurring part in the decimal portion, the recurring decimal has been multiplied by 100 (two zeros at the right side of 1) The second product has been subtracted from the first product. In one side of the result of subtraction is a whole number and at the other side of the result is $(100000 - 100) = 99900$ times of the value of the given recurring decimal. Dividing both the sides by 99900, the required fraction is obtained.

Activity : Express $0.\dot{4}\dot{1}$ and $3.04\dot{6}2\dot{3}$ into fractions.

Rules of Transformation of Recurring Decimals into Simple Fractions

Numerator of the required fraction = the result by subtracting the number obtained from exempting the decimal point of the given decimal point and the non-recurring part.

Denominator of the required fraction = Numbers formed by putting the number of 9 equal to the number of digits in the recurring part of the from the number of zeros equal to the number of digits in the non-recurring part. Here the above rules are directly applied to convert some recurring decimals into simple factions.

Example 10. Express $45.2\dot{3}4\dot{6}$ into simple fraction.

Solution : $45.2\dot{3}4\dot{6} = \frac{452346 - 452}{9990} = \frac{451894}{9990} = \frac{225947}{4995} = 45\frac{1172}{4995}$

Required fraction is $45\frac{1172}{4995}$

Example 11. Express $32.\dot{5}6\dot{7}$ into simple fraction.

Solution : $32.\dot{5}6\dot{7} = \frac{32567 - 32}{999} = \frac{32535}{999} = \frac{3615}{111} = \frac{1205}{37} = 32\frac{21}{37}$

Required fraction is $32\frac{21}{37}$.

Activity : Express $0.0\dot{1}\dot{2}$ and $3.31\dot{2}\dot{4}$ into fraction.

Similar recurring decimals and Non-similar Recurring decimals :

If the numbers of digits in non-recurring part of recurring decimals are equal and also numbers of digits in the recurring parts are equal, those are called similar recurring decimals. Other recurring decimals are called non-similar recurring decimals. For example : $12.\dot{4}\dot{5}$ and $6.\dot{3}\dot{2}$; $9.45\dot{3}$ and $125.89\dot{7}$ are similar recurring decimals. Again, $0.3\dot{4}5\dot{6}$ and $7.45\dot{7}8\dot{9}$; $6.43\dot{5}\dot{7}$ and $2.89\dot{3}4\dot{5}$ are none-similar recurring decimals.

The Rules of Changing Non-Similar Recurring Decimals into Similar Recurring Decimals

The value of any recurring decimals is not changed, if the digits of its recurring part are written again and again, For Example, $6.45\dot{3}\dot{7} = 6.45\dot{3}73\dot{7} = 6.4537\dot{3}\dot{7}$ $= 6.45373\dot{7}\dot{3}$. Here each one is a recurring decimal, $6.45373737.........$ is a non-terminating decimal.

It will be seen that each recurring decimal if converted into a simple fraction has the same value.

$$6.45\dot{3}\dot{7} = \frac{64537-645}{9900} = \frac{63892}{9900}$$

$$6.45\dot{3}73\dot{7} = \frac{6453737-645}{999900} = \frac{6453092}{999900} = \frac{63892}{9900}$$

$$6.4537\dot{3}\dot{7} = \frac{6453737-64537}{990000} = \frac{6389200}{990000} = \frac{63892}{9900}$$

In order to make the recurring decimals similar, number of digits in the non-recurring part of each recurring decimal is to be made equal to the number of digits of non-recurring part of that recurring decimal in which greatest number of digits in the non-recurring part exists and the number of digits in the recurring part of each recurring decimal is also to be made equal to the lowest common multiple of the numbers of digits of recurring parts of recurring decimals.

Example 12. Convert $5.\dot{6}$, $7.3\dot{4}\dot{5}$ and $10 \cdot 78\dot{4}2\dot{3}$ into similar recurring decimals.

Solution : The number of digits of non-recurring part of $5 \cdot \dot{6}$, $7 \cdot 3\dot{4}\dot{5}$ and $10.78\dot{4}2\dot{3}$ are $0, 1$ and 2 respectively. Here the number of dig its in the non-recurring part occurs in $10.78\dot{4}2\dot{3}$ and that number is 2. Therefore to make the recurring decimals similar the number of digits in the non-recurring part of each recurring decimal is to be made 2. Again, the numbers of digits to recurring parts of $5.\dot{6}$, $7.3\dot{4}\dot{5}$ and $10.78\dot{4}2\dot{3}$ are $1, 2$ and 3 respectively. The lowest common multiple of $1, 2$ and 3 is 6. So the number of digits in the recurring part of each recurring decimal would be 6 in order to make them similar.

So, $5.\dot{6} = 5.666\dot{6}666\dot{6}$, $7.3\dot{4}\dot{5} = 7.34\dot{5}4545\dot{4}$ and $10.78\dot{4}2\dot{3} = 10.78\dot{4}23423\dot{3}$

Required similar recurring decimals are $5.666\dot{6}666\dot{6}$, $7.34\dot{5}4545\dot{4}$, $10.78\dot{4}23423\dot{3}$ respectively.

Example 13. Convert 1.7643, $3.\dot{2}\dot{4}$ and $2.78\dot{3}4\dot{6}$ into similar recurring decimals.

Solution : In 1.7643 the number of digits in the non-recurring part means 4 digits after decimal point and here there is no recurring part.

In $3.\dot{2}\dot{4}$ the number of digits in the recurring and non-recurring parts are respectively 0 and 2.

In $2.78\dot{3}4\dot{6}$ the number of digits in the recurring and non-recurring parts are respectively 2 and 3.

The highest number of digits in the nonrecurring parts is 4 and the L.C.M. of the numbers of digits in the recurring parts i.e. 2 and 3 is 6. The numbers of digits is the recurring and nonrecurring parts of each decimal will be respectively 4 and 6.

$\therefore 1.7643 = 1.7643\dot{0}0000\dot{0}$; $3.\dot{2}\dot{4} = 3.2424\dot{2}4242\dot{4}$; $2.78\dot{3}4\dot{6} = 2.7834\dot{6}3463\dot{4}$

Required recurring similar decimals are $1.7643\dot{0}0000\dot{0}$, $3.2424\dot{2}4242\dot{4}$ and $2.7834\dot{6}3463\dot{4}$

Remark : In order to make the terminating fraction similar, the required number of zeros is placed after the digits at the extreme right of decimal point of each decimal fraction. The number of non-recurring decimals and the numbers of digits of non-recurring part of decimals after the decimal points are made equal using recurring digits. After non-recurring part the recurring part can be started from any digit.

Activity : Express 3.467, $2.01\dot{2}\dot{4}\dot{3}$ and $7.52\dot{5}\dot{6}$ into similar recurring fractions.

Addition and Subtraction of Recurring Decimals

In the process of addition or subtraction of recurring decimals, the recurring decimals are to be converted into similar recurring decimals. Then the process of addition or subtraction as that of terminating decimals is followed. If addition or subtraction of terminating decimals and recurring decimals together are done, in order to make recurring decimals similar, the number of digits of non-recurring part of each recurring should be equal to the number of digits between the numbers of digits after the decimal points of terminating decimals and that of the non-recurring parts of recurring decimals. The number of digits of recurring part of each recurring decimal will be equal to L.C.M. as obtained by applying the rules stated earlier and in case of terminating decimals, necessary numbers of zeros are to be used in its recurring parts. Then the same process of addition and subtraction is to be done following the rules of terminating decimals. The sum or the difference obtained in this way will not be the actual one. It should be observed that in the process of addition of similar decimals if any number is to be carried over after adding the digits at the extreme left of the recurring part of the decimals then that number is added to the sum obtained and thus the actual sum is found. In case of subtraction the number to be carried over is to subtract from the difference obtained and thus actual result is found. The sum or difference which is found in this way is the required sum or difference.

Remark (a) : The sum or difference of recurring decimals is also a recurring decimal. In this sum or difference the number of digits in the non-recurring part will be equal to the number of digits in the non-recurring part of that recurring decimal, which have the highest number of digits in its non-recurring part. Similarly, the number of digits in the recurring part of the sum or the result of subtraction will be the equal to L.C.M. of the numbers of digits of recurring parts of recurring decimals. If there is any terminating decimals, the number of digits in the non-recurring part of each recurring decimal will be equal to the highest numbers of digits that occurs after the decimal point.

(b) Converting the recurring decimals into simple fractions, addition and subtraction may be done according to the rules as used in case of simple fractions and the sum or difference is converted into decimal fractions. But this process needs more time.

Example 14. Add : $3\cdot\dot{8}\dot{9}$, $2\cdot1\dot{7}\dot{8}$ and $5\cdot89\dot{7}9\dot{8}$

Solution : Here the number of digits in the non-recurring part will be 2 and the number of digits in the recurring part will be 6 which is L.C.M. of 2,2 and 3.

At first three recurring decimals are made similar.

$$
\begin{array}{ll}
3\cdot\dot{8}\dot{9} & = 3\cdot89\dot{8}9898\dot{9} \\
2\cdot1\dot{7}\dot{8} & = 2\cdot127\dot{8}7878\dot{7} \\
\underline{5\cdot89\dot{7}9\dot{8}} & = \underline{589\dot{7}9879\dot{8}} \\
 & \ 11\cdot97576574 \\
 & \ \underline{\qquad\qquad +2}
\end{array}
$$

[$8+8+7+2=25$, Here 2 is the number to be carried over, 2 of 25 has been added.]

$$11\cdot97\dot{5}7657\dot{6}$$

The required sum is $11\cdot97\dot{5}7657\dot{6}$ or $11\cdot97\dot{5}7\dot{6}$

Remark : In the sum the number in the recurring part is 575675. But the value is not changed if 576 is taken as the number of recurring part.

Note : To make clear the concept of adding 2 at the extreme right side, this addition is done in another method :

$$
\begin{array}{lcl}
3\cdot\dot{8}\dot{9} & = & 3\cdot 89\dot{8}98989\dot{9}\,|\,89 \\
2\cdot 1\dot{7}\dot{8} & = & 2\cdot 17\dot{8}78787\dot{7}\,|\,87 \\
5\cdot 89\dot{7}9\dot{8} & = & 5\cdot 89\dot{7}98798\dot{8}\,|\,79 \\
\hline
 & & 11.97\dot{5}76576\dot{6}\,|\,55
\end{array}
$$

Here the number is extended upto 2 more digits after the completion of the recurring part. The additional digits are separated by drawing a vertical line. Then 2 has been carried over from the sum of the digits at the right side of the vertical line and this 2 is added to the sum of the digit at the left side of the vertical line. The digit in the right side of the vertical line are the same and the recurring point. Therefore both the sums are the same.

Example 15. Add : $8\cdot 9\dot{4}7\dot{8}$, $2\cdot 346$ and $4\cdot\dot{7}\dot{1}$.

Solution : To make the decimals similar, the number of digits of non-recurring parts would be 3 and that of recurring parts would be 6 which is L.C.M. of 3 and 2.

$$
\begin{array}{lcl}
8\cdot 9\dot{4}7\dot{8} & = & 8\cdot 947\dot{8}4784\dot{7} \\
2\cdot 346 & & 2\cdot 346\dot{0}0000\dot{0} \\
4\cdot\dot{7}\dot{1} & = & 4\cdot 717\dot{1}7171\dot{7} \\
\hline
 & & 16\cdot 011019564 \\
 & & +1 \\
\hline
 & & 16\cdot 011\dot{0}1956\dot{5}
\end{array}
$$

[$8+0+1+1=10$, Here the digit in the second place on the left is 1 which is to be carried over. Therefore 1 of 10 is added.]

The required sum is $16\cdot 011\dot{0}1956\dot{5}$.

Activity : Add :1. $2\cdot 0\dot{9}\dot{7}$ and $5\cdot 12\dot{7}6\dot{8}$ 2. $1\cdot 34\dot{5}$, $0\cdot 31\dot{5}7\dot{6}$ and $8\cdot 056\dot{7}\dot{8}$

Example 16. Subtract $5\cdot 24\dot{6}7\dot{3}$ from $8\cdot 2\dot{4}\dot{3}$.

Solution : Here the number of digits in the non-recurring part would be 2 and that of recurring part is 6 which is L.C.M. of 2 and 3. Now making two decimal numbers similar, subtraction is done.

$$
\begin{array}{lcl}
8\cdot 2\dot{4}\dot{3} & = & 8\cdot 24\dot{3}4343\dot{4} \\
5\cdot 24\dot{6}7\dot{3} & = & 5\cdot 24\dot{6}7367\dot{3} \\
\hline
 & & 2\cdot 99669761 \\
 & & -1 \\
\hline
 & & 2\cdot 99\dot{6}69760
\end{array}
$$

[Subtracting 6 from 3, 1 is to be carried over.]

The required sum is $2\cdot 99\dot{6}69760$.

Remark : If the digit at the beginning place of recurring point in the number from which deduction to be made is smaller than that of the digit in the number 1 is to be subtracted from the extreme right hand digit of the result of subtraction.

Note : In order to make the conception clear why 1 is subtracted, subtraction is done in another method as shown below :

$$
\begin{array}{ll}
8\cdot 2\dot{4}\dot{3} & = 8\cdot 24\dot{3}43434\dot{4}\,|\,34 \\
5\cdot 24\dot{6}7\dot{3} & = 5\cdot 24\dot{6}73673\dot{3}\,|\,67 \\
\hline
 & \ 2\cdot 99\dot{6}6976\dot{0}\,|\,67
\end{array}
$$

The required difference is $2\cdot 99\dot{6}6976\dot{0}\,|\,67$

Here both the differences are the same.

Example 17. Subtract $16\cdot\dot{4}3\dot{7}$ from $24\cdot 45\dot{6}4\dot{5}$.

Solution :

$$
\begin{array}{lr}
24\cdot 45\dot{6}4\dot{5} & = 24\cdot 45\dot{6}4\dot{5} \\
16\cdot\dot{4}3\dot{7} & =16\cdot 43\dot{7}4\dot{3} \\
\hline
 & 8\cdot 01902 \\
 & -1 \\
\hline
 & 8\cdot 01\dot{9}0\dot{1}
\end{array}
$$

[7 is subtracted from 6.1 is to be carried over.]

The required difference is $8.01\dot{9}0\dot{1}$

Note :

$$
\begin{array}{lr}
24\cdot 45\dot{6}4\dot{5} & = 24\cdot 45\dot{6}4\dot{5}\,|\,64 \\
16\cdot\dot{4}3\dot{7} & =16\cdot 43\dot{7}4\dot{3}\,|\,74 \\
\hline
 & 8\cdot 01\dot{9}0\dot{1}\,|\,90
\end{array}
$$

Activity : Subtract :1. $10\cdot 418$ from $13\cdot 12\dot{7}8\dot{4}$ 2. $9\cdot 12\dot{6}4\dot{5}$ from $23\cdot 03\dot{9}\dot{4}$

Multiplication and Division of Recurring Decimals :

Converting recurring decimals into simple fraction and completing the process of their multiplication or division, the simple fraction thus obtained when expressed into a decimal fraction will be the product or quotient of the recurring decimals. In the process of multiplication or division amongst terminating and recurring decimals the same method is to be applied. But in case of making division easier if both the divident and the divisior are of recurring decimals, we should convert them into similar recurring decimals.

Example 18. Multiply $4\cdot\dot{3}$ by $5\cdot\dot{7}$.

Solution : $4\cdot\dot{3} = \frac{43-4}{9} = \frac{39}{9} = \frac{13}{3}$

$5\cdot\dot{7} = \frac{57-5}{9} = \frac{52}{9}$

$\therefore\ 4\cdot\dot{3} \times 5\cdot\dot{7} = \frac{13}{3}\times\frac{52}{9} = \frac{676}{27} = 25\cdot\dot{0}3\dot{7}$

The required product is $25\cdot\dot{0}3\dot{7}$

Example 19. Multiply $0\cdot2\dot{8}$ by $42\cdot\dot{1}\dot{8}$.

Solution : $0\cdot2\dot{8} = \frac{28-2}{90} = \frac{26}{90} = \frac{13}{45}$

$42\cdot\dot{1}\dot{8} = \frac{4218-42}{99} = \frac{4176}{99} = \frac{464}{11}$

$= \frac{13}{45}\times\frac{464}{11} = \frac{6032}{495} = 12\cdot1\dot{8}\dot{5}$

The required product is $12\cdot1\dot{8}\dot{5}$

Example 20. $2\cdot5\times4\cdot3\dot{5}\times1\cdot2\dot{3}\dot{4} =\ ?$

Solution : $2\cdot5 = \frac{25}{10} = \frac{5}{2}$

$4\cdot3\dot{5} = \frac{435-43}{90} = \frac{392}{90}$

$1\cdot2\dot{3}\dot{4} = \frac{1234-12}{990} = \frac{1222}{990} = \frac{611}{495}$

$\frac{5}{2}\times\frac{392}{90}\times\frac{611}{495} = \frac{196\times611}{8910} = \frac{119756}{8910} = 13\cdot44062....$

The required product is $13\cdot44062$

Activity : 1. Multiply $1\cdot1\dot{3}$ by $2\cdot6$. 2. $0\cdot\dot{2}\times1\cdot\dot{1}\dot{2}\times0\cdot0\dot{8}\dot{1} =\ ?$

Example 21. Divide $7\cdot\dot{3}\dot{2}$ by $0\cdot2\dot{7}$.

Solution : $7\cdot\dot{3}\dot{2} = \frac{732-7}{99} = \frac{725}{99}$

$0\cdot2\dot{7} = \frac{27-2}{90} = \frac{25}{90} = \frac{5}{18}$

$\therefore\ 7\cdot\dot{3}\dot{2} \div 0\cdot2\dot{7} = \frac{725}{99}\div\frac{5}{18} = \frac{725}{99}\times\frac{18}{5} = \frac{290}{11} = 26\cdot\dot{3}\dot{6}$

The required quotient is $26\cdot\dot{3}\dot{6}$

Example 22. Divide $2\cdot\dot{2}71\dot{8}$ by $1\cdot9\dot{1}\dot{2}$

Solution : $2\cdot\dot{2}71\dot{8} = \dfrac{22718-2}{9999} = \dfrac{22176}{9999}$

$$1\cdot9\dot{1}\dot{2} = \frac{1912-19}{990} = \frac{1893}{990}$$

$$\therefore 2\cdot\dot{2}71\dot{8} \div 1\cdot9\dot{1}\dot{2} = \frac{22716}{9999} \div \frac{1893}{990} = \frac{22716}{9999} \times \frac{990}{1893} = \frac{120}{101} = 1\cdot\dot{1}88\dot{1}$$

The required quotient is $1\cdot\dot{1}88\dot{1}$

Example 23. Divide $9\cdot45$ by $2\cdot8\dot{6}\dot{3}$.

Solution : $9\cdot45 \div 2\cdot8\dot{6}\dot{3} = \dfrac{945}{100} \div \dfrac{2863-28}{990} = \dfrac{945}{100} \times \dfrac{990}{2835}$

$$= \frac{189\times99}{2\times2835} = \frac{33}{10} = 3\cdot3$$

The required quotient is : $3\cdot3$

Remark : Product of recurring decimals and quot ient of recurring decimals may be or may not be a recurring decimal.

Activity : 1. Divide $0\cdot\dot{6}$ by $0\cdot\dot{9}$. 2. Divide $0\cdot7\dot{3}\dot{2}$ by $0\cdot0\dot{2}\dot{7}$

Non Terminating Decimals

There are many decimal fractions in which the number of digits after its decimal point is unlimited, again one or more than one digit does not occur repeatedly. Such decimal fractions are called non-terminating decimal fractions, For example, 5.134248513942307 .... is a non-terminating decimal number. The square root of 2 is a non terminating decimal. Now we want to find the square root of 2.

1	2	1·4142135........
	1	
24	100	
	96	
281	400	
	281	
2824	11900	
	11296	

28282	60400
	56564
282841	383600
	282841
2828423	10075900
	8485269
28284265	159063100
	141421325
	17641775

If the above process is continued for ever, it will never end.

$\therefore\ \sqrt{2} = 1 \cdot 4142135......$ is a non terminating decimal number.

The value upto the definite number of decimal places and the approximate value upto some decimal places.

It is not the same to find the value of nonterminating decimals upto definite number of decimal place and the approximate value into some decimal places. For example, $5 \cdot 4325893......$ upto four decimal places will be $5 \cdot 4325,$ but the approximate value of the decimal, $5 \cdot 4325893....$ upto four decimal places will be $5 \cdot 4326$. Here, the value upto 2 decimal places and the approximate value upto 2 decimal places are the same. This value is $5 \cdot 43$. In this way the approximate terminating decimals can also be found.

Remark : When it is needed to find the value upto some decimal places, the digits that occur in those places are to be written without any alternatives of those digits. If approximate values are to be identified, we should add 1 with the last digit when there is 5, 6, 7, 8 or 9 after the decimal places. But if it is 1, 2, 3 or 4 digits remain unchanged. In this case, correct value upto decimal place or approximate value upto decimal place are almost equal. We should find the value upto 1 place more to the required wanted value.

Example 24. Find the square root of 13 and write down the approximate value upto 3 decimal places.

Solution : 3) 13 ($3 \cdot 605551........$

	9
66	400
	396

7205 | 40000
36025

72105 | 3697500
3605525

7211101 | 9197500
7211101

1986399

$\therefore$ The required square root is $3 \cdot 605551........$.

$\therefore$ The required approximate value upto 3 decimal places is $3 \cdot 606$.

Example 25. Find the value and approximate value of $4 \cdot 4623845.......$ upto $1, 2, 3, 4$ and 5 decimal places.

Solution : The value of $4 \cdot 4623845$ upto 1 decimal place is $4 \cdot 4$

and approximate value 1 ,, ,, ,, $4 \cdot 5$

Value upto 2 ,, ,, ,, $4 \cdot 46$

and approximate value upto 2 ,, ,, ,, $4 \cdot 46$

Value upto 3 ,, ,, ,, $4 \cdot 462$

and approximate value upto 3 ,, ,, ,, $4 \cdot 462$

Value upto 4 ,, $4 \cdot 4623$

and approximate value upto 4 ,, ,, ,, $4 \cdot 4624$

Value upto 5 ,, ,, ,, $4 \cdot 46238$

and approximate value upto 5 ,, ,, ,, $4 \cdot 46238$

Activity : Find the square root of 29 and find the value upto two decimal places and the approximate value upto two decimal place.

Exercise 1

1. Prove that, (a) $\sqrt{5}$ (b) $\sqrt{7}$ (c) $\sqrt{10}$ is an irrational number.
2. (a) Find the two irrational numbers between 0.31 and 0.12.

   (b) Find a rational and an irrational numbers between $\frac{1}{\sqrt{2}}$ and $\sqrt{2}$.
3. (a) Prove that, square of any odd integer number is an odd number.

   (b) Prove that, the product of two consecutive even numbers is divisible by 8.

4. Express into recurring decimal fractions :(a) $\frac{1}{6}$ (b) $\frac{7}{11}$ (c) $3\frac{2}{9}$ (d) $3\frac{8}{15}$

5. Express into simple fractions :

(a) $0\cdot\dot{2}$ (b) $0\cdot\dot{3}\dot{5}$ (c) $0\cdot1\dot{3}$ (d) $3\cdot7\dot{8}$ (e) $6\cdot2\dot{3}0\dot{9}$

6. Express into similar recurring fractions :

(a) $2\cdot\dot{3},\ 5\cdot2\dot{3}\dot{5}$ (b) $7\cdot2\dot{6},\ 4\cdot23\dot{7}$

(c) $5\cdot\dot{7},\ 8\cdot\dot{3}\dot{4},\ 6\cdot\dot{2}4\dot{5}$ (d) $12\cdot32,\ 2\cdot1\dot{9},\ 4\cdot32\dot{5}\dot{6}$

7. Add :(a) $0\cdot4\dot{5}+0\cdot13\dot{4}$ (b) $2\cdot0\dot{5}+8\cdot0\dot{4}+7\cdot018$ (c) $0\cdot00\dot{6}+0\cdot\dot{9}\dot{2}+0\cdot0\dot{1}3\dot{4}$

8. Subtract :

(a) $3\cdot\dot{4}-2\cdot1\dot{3}$ (b) $5\cdot\dot{1}\dot{2}-3\cdot4\dot{5}$

(c) $8\cdot49-5\cdot3\dot{5}\dot{6}$ (d) $19\cdot34\dot{5}-13\cdot2\dot{3}4\dot{9}$

9. Multiply:

(a) $0\cdot\dot{3}\times0\cdot\dot{6}$ (b) $2\cdot\dot{4}\times0\cdot\dot{8}\dot{1}$ (c) $0\cdot6\dot{2}\times0\cdot\dot{3}$ (d) $42\cdot\dot{1}\dot{8}\times0\cdot2\dot{8}$

10. Divide :

(a) $0\cdot\dot{3}\div0\cdot\dot{6}$ (b) $0\cdot3\dot{5}\div1\cdot\dot{7}$ (c) $2\cdot3\dot{7}\div0\cdot4\dot{5}$ (d) $1\cdot\dot{1}8\dot{5}\div0\cdot\dot{2}\dot{4}$

11. Find the root (upto three decimal places) and write down the approximate values of the square roots upto two decimal places :

(a) 12 (b) $0\cdot\dot{2}\dot{5}$ (c) $1\cdot3\dot{4}$ (d) $5\cdot1\dot{3}0\dot{2}$

12. Find the rational and irrational numbers from the following numbers :

(a) $0\cdot\dot{4}$ (b) $\sqrt{9}$ (c) $\sqrt{11}$ (d) $\frac{\sqrt{6}}{3}$ (e) $\frac{\sqrt{8}}{\sqrt{7}}$ (f) $\frac{\sqrt{27}}{\sqrt{48}}$ (g) $\frac{\frac{2}{3}}{\frac{3}{7}}$ (h) $5\cdot\dot{6}3\dot{9}$

13. Simplify :

(a) $(0\cdot\dot{3}\times0\cdot8\dot{3})\div(0\cdot5\times0\cdot\dot{1})+0\cdot3\dot{5}\div0\cdot0\dot{8}$

(b) $[(6\cdot27\times0\cdot5)\div\{(0\cdot5\times0\cdot75)\times8\cdot36\}]$
$\div\{(0\cdot25\times0\cdot1)\times(0\cdot75\times21\cdot\dot{3})\times0\cdot5\}$

14. $\sqrt{5}$ and 4 are two real numbers.

(a) Which one is rational and which one is irrational.

(b) Find the two irrational numbers between $\sqrt{5}$ and 4.

(c) Pove That, $\sqrt{5}$ is an irrational number.

Chapter Two

Set and Function

The word 'set' is familiar to us, such as dinner set, set of natural numbers, set of rational numbers etc. As a modern weapon of mathematics, the use of set is extensive. The German mathematician George Cantor (1844 – 1918) first explained his opinion about set. He created a sensation in mathematics by expressing the idea of infinite set and his conception of set is known as 'set theory'. In this chapter, the main objectives are to solve problems through using mathematics and symbols from the conception of set and to acquire knowledge about function.

At the end of this chapter, the students will be able to :

- Explain the conception of set and subset and express them by symbols
- Describe the method of expressing set
- Explain the infinite set and differentiate between finite and infinite set
- Explain and examine the union and the intersection of set
- Explain power set and form power set with two or three elements
- Explain ordered pair and cartesian product
- Prove the easy rules of set by example and Venn Diagram and solve various problems using the rules of set operation
- Explain and form sets and functions
- Explain what are domain and range
- Determine the domain and range of a function
- Draw the graph of the function.

Set

Well defined assembling or collection of objects of real or imaginative world is called sets, such as, the set of three textbook of Bangla, English and Mathematics, set of first ten natural odd numbers, set of integers, set of real numbers etc.

Set is generally expressed by the capital letters of english alphabets, $A, B, C, ...X, Y, Z$. For example, the set of three numbers 2, 4, 6 is $A = \{2, 4, 6\}$

Each object or member of set is called set element. Such as, if $B = \{a, b\}$, a and b are elements of B. The sign of expressing an element is '$\in$'.

$\therefore$ $a \in B$ and read as a belongs to B

$b \in B$ and read as b belongs to B

no element c is in the above set B.

$\therefore$ $c \notin B$ and read as c does not belong to B.

Method of describing Sets :

Set is expressed in two methods :(1) Roster Method or Tabular Method and (2) Set Builder Method.

(1) Tabular Method : In this methods, all the set elements are mentioned particularly by enclosing them within second bracket { } and if there is more than one element, the elements are separated by using a comma (,).

For example : $A = \{a, b\}$, $B = \{2, 4, 6\}$, $C =$ {Niloy, Tisha, Shuvra} etc.

(2) Set Builder Method : In this methods, general properties are given to determine the set element, without mentioning them particularly :

Such as, $A = \{x : x$ is a natural odd number}, $B = \{x : x$ denotes the first five students of class IX} etc.

Here, by ':' 'such as' or in short 'such that' is indicated. Since in this method, set rule or condition is given to determining the set elements of this method, is called rule method.

Example 1. Express the set $A = \{7, 14, 21, 28\}$ by set builder method.

Solution : The elements of set A are $7, 14, 21, 28$

Here, each element is divisible by 7, that is, multiple of 7 and not greater than 28.

$\therefore$ $A = \{x : x$ multiple of 7 and $x \leq 28\}$.

Example 2. Express the set $B = \{x : x,$ factors of 28} by tabular method.

Solution : Here, $28 = 1 \times 28$

$= 2 \times 14$

$= 4 \times 7$

$\therefore$ factors of 28 are $1, 2, 4, 7, 14, 28$

Required set $B = \{1, 2, 4, 7, 14, 28\}$

Example 3. Express $C = \{x : x$ is a positive integer and $x^2 < 18\}$ by tabular method.

Solution : Positive integers are $1, 2, 3, 4, 5, ...$

Here if $x = 1$, $x^2 = 1^2 = 1$

if $x = 2$, $x^2 = 2^2 = 4$

if $x = 3$, $x^2 = 3^2 = 9$

if $x = 4$, $x^2 = 4^2 = 16$

if $x = 5$, $x^2 = 5^2 = 25$; which is greater than 18.

$\therefore$ The acceptable positive integers by the condition are $1, 2, 3, 4$

$\therefore$ Required set is $C = \{1, 2, 3, 4\}$

> **Activity :** 1. Express the set $C = \{-9, -6, -3, 3, 6, 9\}$ by set builder method.
>
> 2. Express the set $Q = \{y : y$ is an integer and $y^3 \leq 27\}$ by tabular method.

Finite set : The set whose numbers of elements can be determined by counting is called finite set. For example, D = {*x, y, z*}, E = {3, 6, 9,.......,60}, $F = \{x : x$ is a prime number and $30 < x < 70\}$ etc. are finite set. Here D set has 3 elements, E set has 20 elements and F set has 9 elements.

Infinite set : The set whose numbers of elements can not be determined by counting is called infinite set. For example :, $A = \{x : x$ is natural odd numbers$\}$, set of natural number $N = \{1, 2, 3, 4, ...\}$, set of integers $Z = \{..... -3, -2, -1, 0, 1, 2, 3.......\}$, set of rational numbers $Q = \left\{\frac{p}{q} : p\right.$ and q is as integer and $q \neq 0\}$, set of real numbers = R etc. are infinite set.

Example 4. Show that the set of all natural numbers is an infinite set.

Solution : Set of natural number $N = \{1, 2, 3, 4, 5, 6, 7, 8, ...\}$

Taking odd natural numbers from set N, the formed set $A = \{1, 3, 5, 7, ...\}$

,, even ,, ,, ,, ,, N, the formed set $B = \{2, 4, 6, 8, ...\}$

The set of multiple of 3 $C = \{3, 6, 9, 12, ...\}$ etc.

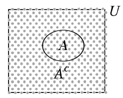
Here, the elements of the set A, B, C formed from set N can not be determined by counting. So A, B, C is an infinite set.

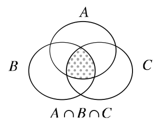
$\therefore$ N is an infinite set.

> **Activity :** Write the finite and infinite set from the sets given below :
>
> 1. $\{3, 5, 7\}$ 2. $\{1, 2, 2^2, ...2^{10}\}$ 3. $\{3, 3^2, 3^3, ...\}$ 4. $\{x : x$ is an integer and $x < 4\}$
>
> 5. $\left\{\frac{p}{q} : p\right.$ and q are coprime and $q > 1\}$ 6. $\{y : y \in N$ and $y^2 < 100 < y^3\}$.

Empty set : The set which has no element is called empty set. Empty set is expressed by {} or ϕ. Such as, set of three male students of Holycross school $\{x \in N : 10 < x < 11\}$, $\{x \in N : x$ is a prime number and $23 < x < 29\}$ etc.

Venn-Diagram : John Venn (1834-1883) introduced set activities by diagram. Here the geometrical figure on the plane like rectangular area, circular area and triangular area are used to represent the set under consideration. These diagrams are named Venn diagram after his name.





Subset : $A = \{a, b\}$ is a set. By the elements of set A, the sets $\{a, b\}$, $\{a\}$, $\{b\}$ can be formed. Again, by not taking any element ϕ set can be formed.

Here, each of $\{a, b\}$, $\{a\}$, $\{b\}$, ϕ is subset of set A.

So, the number of sets which can be formed from any set is called subset of that set. The sign of subset is $\subset$. If B is the subset of A, it is read as $B \subset A$. B is a subset of A. From the above subsets, $\{a, b\}$ set is equal to A.

$\therefore$ Each set is the subset of itself.

Again, from any set, ϕ set can be formed.

$\therefore$ ϕ is a subset of any set.

$Q = \{1, 2, 3\}$ and $R = \{1, 3\}$ are two subsets of $P = \{1, 2, 3\}$. Again $P = Q$

$\therefore$ $Q \subseteq P$ and $R \subset P$.

Proper Subset :

If the number of elements of any subset formed from a set is less than the given set, it is called the proper subset. For example : $A = \{3, 4, 5, 6\}$ and $B = \{3, 5\}$ àre two sets. Here, all the elements of B exist in set A. $\therefore$ $B \subset A$

Again, the number of elements of B is less then the number of elements of A.

$\therefore$ B is a proper subset of A and expressed as $B \subseteq A$.

Example 5. Write the subsets of $P = \{x, y, z\}$ and find the proper subset from the subsets.

Solution : Given, $P = \{x, y, z\}$

Subsets of P are $\{x, y, z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x\}$, $\{y\}$, $\{z\}$, ϕ.

Proper subsets of P are $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x\}$, $\{y\}$, $\{z\}$

Equivalent Set :

If the elements of two or more sets are the same, they are called equivalent sets. Such as, $A = \{3, 5, 7\}$ and $B = \{5, 3, 7\}$ are two equal sets and written as $A = B$.

Again, if $A = \{3, 5, 7\}$, $B = \{5, 3, 3, 7\}$ and $C = \{7, 7, 3, 5, 5\}$, the sets A, B and C are equivalent. That is, $A = B = C$

It is to be noted if the order of elements is changed or if the elements are repeated, there will be no change of the set.

Difference of Set : Suppose, $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5\}$. If the elements of set B are discarded from set A, the set thus formed is $\{1, 2, 4\}$ and written as $A \setminus B$ or $A - B = \{1, 2, 3, 4, 5\} - \{3, 5\} = \{1, 2, 4\}$

So, if a set is discarded from any set, the set thus formed is called different set.

Example 6. If $P = \{x : x$, factors of 12$\}$and $Q = \{x : x$, multiples of 3 and $x \leq 12\}$, find $P - Q$.

Solution : Given, $P = \{x : x$, factors of 12$\}$

Here, factors of 12 are $1, 2, 3, 4, 6, 12$

$\therefore\ P = \{1, 2, 3, 4, 6, 12\}$

Again, $Q = \{x : x$, multiple of 3 and $x \leq 12\}$

Here, multiple of 3 upto 12 are $3, 6, 9, 12$

$\therefore\ Q = \{3, 6, 9, 12\}$

$\therefore\ P - Q = \{1, 2, 3, 4, 6, 12) - \{3, 6, 9, 12\} = \{1, 2, 4\}$

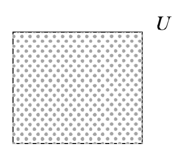
Required set $\{1, 2, 4\}$

Universal Set :

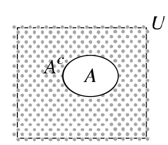
All sets related to the discussions are subset of a definite set. Such as, $A = \{x, y\}$ is a subset of $B = \{x, y, z\}$. Here, set B is called the universal set in with respect to the set A.

So, if all the sets under discussion are subsets of a particular set, that particular set is called the universal set with respect to its subsets.

Universal set is generally expressed by U. But universal set can be expressed by other symbols too. Such as, if set of all even natural numbers $C = \{2, 4, 6........\}$ and set of all natural numbers $N = \{1, 2, 3, 4,...........\}$, the universal set with respect to C will be N.



Complement of a Set :



U is an universal set and A is the subset of U. The set formed by all the elements excluding the elements of set A is called the complement of set A. The complement of the set A is expressed by A^c or A'. Mathematically, $A^c = U \setminus A$

Let, P and Q are two sets and the elements of Q which are not elements of P are called complement set of Q with respect to P and written as $Q^c = P \setminus Q$.

Example 7. If $U = \{1, 2, 3, 4, 6, 7\}$, $A = \{2, 4, 6, 7\}$ and $B = \{1, 3, 5\}$, determine A^c and B^c.

Solution : $A^c = U \setminus A = \{1, 2, 3, 4, 6, 7\} \setminus \{2, 4, 6, 7\} = \{1, 3, 5\}$

and $B^c = U \setminus B = \{1, 2, 3, 4, 6, 7\} \setminus \{1, 3, 5\} = \{2, 4, 6, 7\}$

Required set $A^c = \{1, 3, 5\}$ and $B^c = \{2, 4, 6, 7\}$

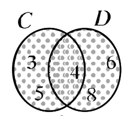
Union of Sets :

The set formed by taking all the elements of two or more sets is called union of sets. Let, A and B are two sets. The union of A and B set is expressed by $A \cup B$ and read as A union B. In the set builder method $A \cup B = \{x : x \in A$ or $x \in B\}$.

Example 8. If $C = \{3, 4, 5\}$ and $D = \{4, 6, 8\}$, determine $C \cup D$.

Solution : Given that, $C = \{3, 4, 5\}$ and $D = \{4, 6, 8\}$

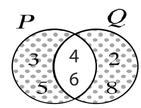
$\therefore$ $C \cup D = \{3, 4, 5\} \cup \{4, 6, 8\} = \{3, 4, 5, 6, 8\}$



$C \cup D$

Intersection of Sets:

The set formed by the common elements of two or more sets is called intersection of sets. Let, A and B are two sets. The intersection of A and B is expressed by $A \cap B$ and read as A intersection B. In set building method, $A \cap B = \{x : x \in A$ and $x \in B\}$.



$P \cap Q$

Example 9. If $P = \{x \in N : 2 < x \leq 6\}$ and $Q = \{x \in N : x$ are even numbers and $x \leq 8\}$, find $P \cap Q$.

Solution : Given that, $P = \{x \in N : 2 < x \leq 6\} = \{3, 4, 5, 6\}$

and $Q = \{x \in N : x$ are even numbers and $x \leq 8\} = \{2, 4, 6, 8\}$

$\therefore$ $P \cap Q = \{3, 4, 5, 6\} \cap \{2, 4, 6, 8\} = \{4, 6\}$

Required set is $\{4, 6\}$

Disjoint Sets:

If there is no common element in between two sets, the sets are disjoined sets. Let, A and B are two sets. If $A \cap B = \phi$, A and B will be mutually disjoint sets.

Activity : If $U = \{1, 3, 5, 7, 9, 11\}$, $E = \{1, 5, 9\}$ and $F = \{3, 7, 11\}$, find, $E^c \cup F^c$ and $E^c \cap F^c$.

Power Sets :

$A = \{m, n\}$ is a set. The subsets of A are $\{m, n\}$, $\{m\}$, $\{n\}$, ϕ. Here, the set of subsets $\{\{m, n\}, \{m\}, \{n\}, \phi\}$ is called power set of set A. The power set of A is expressed as $P(A)$. So, the set formed with all the subsets of any set is called the power set of that set.

Example 10. $A=\{\}$, $B=\{a\}$, $C=\{a,b\}$ are three sets.

Here, $P(A)=\{\phi\}$

$\therefore$ The number of elements of set A is 0 and the number of elements of its power set $=1=2^0$

Again, $P(B)=\{\{a\},\phi\}$

$\therefore$ The number of elements of set B is 1 and the number of elements of its power set is $=2=2^1$

and $P(C)=\{\{a,b\},\{a\},\{b\},\phi\}$

$\therefore$ The number of elements of set C is 2 and the number of elements of its power set is $=4=2^2$

So, if the number of elements is n of a set, the number of elements of its power set will be 2^n.

Activity : If $G=\{1,2,3\}$, determine $P(G)$ and show that the number of elements of $P(G)$ supports 2^n.

Ordered pair :

Amena and Sumona of class VIII stood 1st and 2nd respectively in the merit list in the final examination. According to merit they can be written (Amena, Sumona) as a pair. In this way, the pair fixed is an ordered pair.

Hence, in the pair of elements, the first and second places are fixed for the elements of the pair and such expression of pair is called ordered pair.

If the first element of an ordered pair is x and second element is y, the ordered pair will be (x,y). The ordered pair (x,y) and (a,b) will be equal or $(x,y)=(a,b)$ if $x=a$ and $y=b$.

Example 11. If $(2x+y,3)=(6,x-y)$, find (x,y).

Solution : Given that, $(2x+y,3)=(6,x-y)$

According to the definition of ordered pair, $2x+y=6$........(1)

and $x-y=3$.........(2)

Adding equation (1) and (2), we get, $3x=9$ or, $y=3$

Putting the value of x in equation (1), we get, $6+y=6$ or, $y=0$

$\therefore$ $(x,y)=(3,0)$.

Cartesian Product :

Wangsu decided to give layer of white or blue colour in room of his house at the inner side and red or yellow or green colour at the outer side. The set of colour of inner wall $A=$ {white, blue} and set of colour of outer wall $B=$ {red, yellow, green} Wangsu can apply the colour of his room in the form of ordered pair as (white, red), (white, yellow), (white, green), (blue, red), (blue, yellow), (blue, green). The given ordered pair is written as.

$A\times B=\{$(white, red), (white, yellow), (white, green), (blue, red), (blue, yellow), (blue, green)$\}$

This is the cartesian product set.

In set builder method, $A\times B=\{(x, y); x\in A$ and $y\in B\}$

$A\times B$ is read as A *cross* B.

Example 12. If $P=\{1, 2, 3\}$, $Q=\{3, 4\}$ and $R=P\cap Q$, determine $P\times R$ and $R\times Q$.

Solution : Given that, $P=\{1, 2, 3\}$, $Q=\{3, 4\}$

and $R=P\cap Q=\{1, 2, 3\}\cap\{3, 4\}=\{3\}$

$\therefore\ P\times R=\{1,2,3\}\times\{3\}=\{(1, 3).(2, 3),(3, 3)\}$

and $R\times Q=\{3\}\times\{3, 4\}=\{(3, 3),(3, 4)\}$

> **Activity :** 1. If $\left(\frac{x}{2}+\frac{y}{3}, 1\right)=\left(1, \frac{x}{3}+\frac{y}{2}\right)$, find (x, y).
>
> 2. If $P=\{1, 2, 3\}$, $Q=\{3, 4\}$ and $R=\{x, y\}$, find $(P\cap Q)\times R$ and $(P\cap Q)\times Q$.

Example 13. Find the set where 23 is reminders in each case when 311 and 419 are divided by the natural numbers.

Solution : The numbers when 311 and 419 are divided by rational numbers and 23 is reminder, will be greater than 23 and will be common factors of 311 – 23 = 288 and 419 – 23 = 396.

Let, the set of factors of 288 greater than 23 is A and the set of factors of 396 is B.

Here,

$288=1\times 288=2\times 144=3\times 96=4\times 72=6\times 48=8\times 36=9\times 32=12\times 24=16\times 18$

$\therefore\ A=\{24, 32, 36, 48, 72, 96, 144, 288\}$

Again,

$396=1\times 396=2\times 198=3\times 132=4\times 99=6\times 66=9\times 44=11\times 36=12\times 33=18\times 22$

$\therefore\ B=\{33, 36, 44, 66, 99, 132, 198, 396\}$

$\therefore\ A\cap B=\{24, 32, 36, 48, 72, 96, 144, 288\}\cap\{33, 36, 44, 66, 99, 132, 198, 396\}=\{36\}$

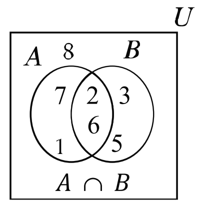
Required set is $\{36\}$

Example 14. If $U=\{1, 2, 3, 4, 5, 6, 7, 8\}$, $A=\{1, 2, 6, 7\}$, $B=\{2, 3, 5, 6\}$ and $C=\{4, 5, 6, 7\}$, show that, (i) $(A\cup B)'=A'\cap B'$ and (ii) $(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$

Solution : (i)

In the figure, U by rectangle and the sets of A and B are denoted by two mutually intersecting circle sets.

set	Elements
$A \cup B$	1, 2, 3, 4, 5, 6, 7
$(A \cup B)'$	4, 8
A'	3, 4, 5, 8
B'	1, 4, 7, 8
$A' \cap B'$	4, 8

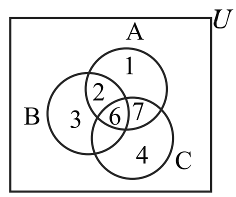


$\therefore\ (A \cup B)' = A' \cap B'$

Solution : (ii) In figure, U by rectangle and sets of A, B, C are denoted by three mutually intersecting circles.

Observe ,

Set	Elements
$A \cap B$	2, 6
$(A \cap B) \cup C$	2, 4, 5, 6, 7
$A \cup C$	1, 2, 4, 5, 6, 7
$B \cup C$	2, 3, 4, 5, 6, 7
$(A \cap C) \cap (B \cup C)$	2, 4, 5, 6, 7

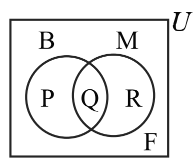


$\therefore\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Example 15. Among 100 students, 92 in Bangla, 80 in Math and 70 have passed in both subjects in any exam. Express the information by Venn diagram and find how many students failed in both subjects.

Solution : In the Venn diagram, the rectangular region denotes set U of 100 students. and the set of passed students in Bangla and Math are denoted by B and M. So, the Venn diagram is divided into four disjoint sets which are denoted by P, Q, R, F.

Here, the set of passed students in both subjects $Q = B \cap M$ whose numbers of elements are 70.



$P =$ the set of passed students in Bangla only, whose number of element $= 92 - 70 = 18$

$R =$ the set of passed student in Math only, whose number of elements $= 80 - 70 = 10$

$P \cup Q \cup R = B \cup M$, the set of passed students in one and both subjects, whose number of elements $= 18 + 10 + 70 = 98$

$F =$ the set of students who failed in both subjects, whose number of elements $= 100 - 98 = 2$

$\therefore$ 2 students failed in both subjects.

Exercise 2.1

1. Express the following sets in tabular method :
   (a) $\{x \in N : x^2 > 9 \text{ and } x^3 < 130\}$
   (b) $\{x \in Z : x^2 > 5 \text{ and } x^3 \le 36\}$
   (c) $\{x \in N : x, \text{ factors of 36 and multiple of 6}\}$
   (d) $\{x \in N : x^3 < 25 \text{ and } x^4 < 264\}$
2. Express the following sets in set builder method :
   (a) $\{3, 5, 7, 9, 11\}$ (b) $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
   (c) $\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$ (d) $\{\pm 4, \pm 5, \pm 6\}$
3. If $A = \{2, 3, 4\}$, $B = \{1, 2, a\}$ and $C = \{2, a, b\}$, determine the sets given below:
   (a) $B \setminus C$ (b) $A \cup B$ (c) $A \cap C$ (d) $A \cup (B \cap C)$ (e) $A \cap (B \cup C)$
4. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, justify the followings :
   (*i*) $(A \cup B)' = A' \cap B'$ (*ii*) $(B \cap C)' = B' \cup C'$
   (*iii*) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (*iv*) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
5. If $Q = \{x, y\}$ and $R = \{m, n, l\}$, find $P(Q)$ and $P(R)$.
6. If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = A \cup B$, show that the number of elements of $P(C)$ is 2^n, where n is the number of element of C.
7. (a) If $(x-1, y+2) = (y-2, 2x+1)$, find the value of x and y.
   (b) If $(ax - cy, a^2 - c^2) = (0, ay - cx)$, find the value of (x, y).
   (c) If $(6x - y, 13) = (1, 3x + 2y)$, find the value of (x, y).
8. (a) If $P = \{a\}$, $Q = \{b, c\}$ then, find $P \times Q$ and $Q \times P$.
   (b) If $A = \{3, 4, 5\}$, $B = \{4, 5, 6\}$ and $C = \{x, y\}$, find $(A \cap B) \times C$.
   (c) If $P = \{3, 5, 7\}$, $Q = \{5, 7\}$ and $R = P \setminus Q$, find $(P \cup Q) \times R$.
9. If A and B are the sets of all factors of 35 and 45 respectively, find $A \cup B$ and $A \cap B$.
10. Find the set of the number where 31 is the remainder in each case when 346 and 556 are divided by natural numbers.
11. Out of 30 students of any class, 20 students like football and 15 students like cricket. The number of students who like any one of the two is 10. Show with the help of Venn diagram, the number of students who do not like two of the sports.
12. Out of 100 students in any exam, 65% in Bangla, 48% in both Bangla and English have passed and 15% have failed in both subjects.

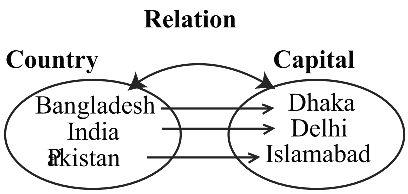
(a) Express the above information by Venn diagram along with brief description.

(b) Find the numbers who have passed only in Bangla and English.

(c) Find the union of two sets of the prime factors of the numbers who have passed and failed in both subjects.

Relation

We know, the capital of Bangladesh is Dhaka and that of India is Delhi and Pakistan is Islamabad. Here, there is a relation of capital with the country. The relation is country-capital relations. The above relation can be shown in set as follows:



That is, country-capital relation = {(Bangladesh, Dhaka), (India, Delhi), (Pakistan, Islamabad)}

If A and B are two sets, the non-zero subset of R of the Cartesian product $A \times B$ of the sets is called relation of B from A.

Here, R is a subset of $A \times B$ set, that is, $R \subseteq A \times B$.

Example 15. Suppose, $A = \{3, 5\}$ and $B = \{2, 4\}$

$\therefore$ $A \times B = \{3, 5\} \times \{2, 4\} = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$

$\therefore$ $R = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$

If the condition is $x > y$, $R = \{(3, 2), (5, 2), (5, 4)\}$

and if the condition is $x < y$, $R = \{3, 4\}$

If an element of set A is x and that of the set B is y and $(x, y) \in R$, we write $x\,R\,y$ and read as x *is related to* y. That the element x is R related to element y. Again, if the relation of a set, from set A that is $R \subseteq A \times A$, R is called A related. So, if the relation is given between set A and B, non-zero subset of ordered pair (x, y) with $y \in B$ related to $x \in A$, is a relation.

Example 16. If $P = \{2, 3, 4\}$, $Q = \{4, 6\}$ and $y = 2x$ is relation under consideration between the elements of P and Q, find the relation.

Solution : Given that, $P = \{2, 3, 4\}$ and $Q = \{4, 6\}$

According to the question, $R = \{(x, y) : x \in P,\ y \in Q$ and $y = 2x\}$

Here, $P \times Q = \{2, 3, 4\} \times \{4, 6\} = \{(2, 4), (2, 6), (3,4)(3, 6), (4, 4), (4, 6)\}$

$\therefore$ $R = \{(2, 4), (3, 6)\}$

Required relation is $\{(2, 4), (3, 6)\}$

Example 17. If $A = \{1, 2, 3\}$, $B = \{0, 2, 4\}$ and the relation $x = y - 1$ is under consideration between elements of C and D, find the relation.

Solution : Given that, $A = \{1, 2, 3\}$, $B = \{0, 2, 4\}$

According to the question, relation $R = \{(x, y) : x \in A,\ y \in B \text{ and } x = y - 1\}$

Here, $A \times B = \{1, 2, 3\} \times \{0, 2, 4\}$

$= \{(1, 0), (1, 2), (1, 4), (2, 0), (2, 2), (2, 4), (3, 0), (3, 2), (3, 4)\}$

$\therefore\ R = \{(1, 2), (3, 4)\}$

> **Activity :** If $C = \{2,5,6\}$, $D = \{4,5\}$ and the relation $x \le y$ is under consideration between elements of C and D, find the relation.

Functions

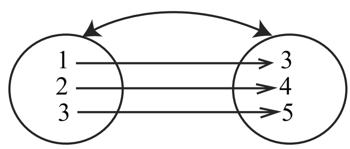
Let us observe the relation between sets A and B below :

Here, When $y = x + 2$,

$y = 3$ for $x = 1$

$y = 4$ for $x = 2$

$y = 5$ for $x = 3$



That is, for each value of x, only one value of y is obtained and the relation between x and y is made by $y = x + 2$. Hence two variable x and y are so related that for any value of x, only one value of y is obtained even y is called the function of x. The function of x is generally expressed by y, $f(x)$, $g(x)$, $F(x)$ etc.

Let, $y = x^2 - 2x + 3$ is a function. Here, for any single value of x, only one value of y is obtained. Here, both x and y are variables but the value of y depends on the value of x. So, x is independent variable and y is dependent variable.

Example 18. If $f(x) = x^2 - 4x + 3$, find $f(-1)$.

Solution : Given that, $f(x) = x^2 - 4x + 3$

$\therefore\ f(-1) = (-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$

Example 19. If $g(x) = x^3 + ax^2 - 3x - 6$, for what value of a will be $g(-2) = 0$?

Solution : Given that, $g(x) = x^3 + ax^2 - 3x - 6$

$\therefore\ g(-2) = (-2)^3 + a(-2)^2 - 3(-2) - 6$

$= -8 + 4a + 6 - 6$

$= -8 + 4a = 4a - 8$

But $g(-2) = 0$

$\therefore\ 4a-8=0$
or, $4a=8$
or, $a=2$
$\therefore$ if $a=2$, $g(-2)=0$.

Domain and Range

The first set of elements of the ordered pair of any relation is called its domain and the set of second elements is called its range.

Let R from set A to set B be a relation, that is, $R \subseteq A\times B$. The set of first elements included in the ordered pair of R will be domain of R and the set of second elements will be range of R. The domains of R is expressed as Dom R and range is expressed as Range R.

Example 20. Relation $S=\{(2,1),(2,2),(3,2),(4,5)\}$. Find the domain and range of the relation.

Solution : Given that, $S=\{(2,1),(2,2),(3,2),(4,5)\}$

In the relation S, the first elements of ordered pair are $2,2,3,4$ and second elements are $1,2,2,5$.

$\therefore$ Dom $S=\{2,3,4\}$ and Range $S=\{1,2,5\}$

Example 21. If $A=\{0,1,2,3\}$ and $R=\{(x,y): x\in A, y\in A$ and $y=x+1\}$, express R in tabular method and determine Dom R and Range R.

Solution : Given that, $A=\{0,1,2,3\}$ and $R=\{(x,y): x\in A, y\in A$ and $y=x+1\}$

From the stated conditions of R we get, $y=x+1$

Now, for each $x\in A$ we find the value of $y=x+1$.

x	0	1	2	3
y	1	2	3	4

Since $4\notin A$, $(3,4)\notin R$

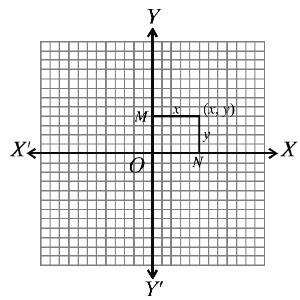
$\therefore\ R=\{(0,1),(1,2),(2,3)\}$

Dom $R=\{0,1,2\}$ and Range $R=\{1,2,3\}$

Activity :
1. If $S=\{(-3,8),(-2,3),(-1,0),(0,-1),(1,0),(2,3)\}$, find domain and range of S.
2. If $S=\{x,y): x\in A, y\in A$ and $y-x=1\}$, where $A=\{-3,-2,-1,0\}$, find Dom. S and Range S.

Graphs :

The diagrammatic view of function is called graphs. In order to make the idea of function clear, the importance of graph is immense. French philosopher and mathematician René Descartes (1596 −1650) at first played a vital role in establishing a relation between algebra and geometry. He introduced the modern way to coplanar geometry by defining the position of point on a plane by two intersecting perpendicular function. He defined the two intersecting perpendicular lines as axes and called the point of intersection origin. On any plane, two intersecting perpendicular straight lines XOX' and YOY' are drawn. The position of any point on this plane can be completely known by these lines. Each of these straight lines are called axis. Horizontal line XOX' is called x axis, Perpendicular line YOY' is called y axis and the point of intersection of the two axes O is called origin.



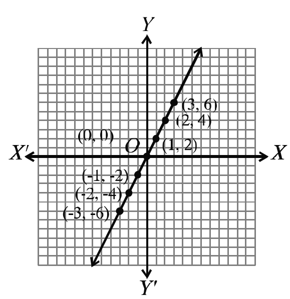
The number with proper signs of the perpendicular distances of a point in the plane from the two axis are called the Coordinates of that point. Let P be any point on the plane between the two axes. PM and PN are drawn perpendicular from the point P to XOX' and YOY' axes respectively. As a result, $PM = ON$ is the perpendicular distance of the point P from YOY' and $PN = OM$ is the perpendicular distance of P from the XOX'. If $PM = x$ and $PN = y$, the coordinates of the point P is (x, y). Here, x is called abscissa or x coordinate and y is called ordinate or y coordinate.

In Cartesian coordinate, the geometrical figure is shown easily. For this reason, generally we put the independent value along the x axis and the dependent value along the y axis. For some values of independent variables from the domain, we find the similar values of dependent variables and form ordered pair to draw the graphs of the function $y = f(x)$. Then place the ordered pair under (x, y) and joint the obtained points in freehands which is the graph of the function $y = f(x)$.

Example 22. Draw the graph of the function $y = 2x$; where $-3 \leq x \leq 3$.

Solution : In the domain $-3 \leq x \leq 3$, for some values of x, we determine some values of y and form a table :

x	-3	-2	-1	0	1	2	3	
y	-6	4	-2	0	2	4	6	



On the graph paper, taking the length of square as unit, we identify points of the table on the place and joint the points in free hand.

Example 23. If $f(x)=\dfrac{3x+1}{3x-1}$, find the value of $\dfrac{f\left(\frac{1}{x}\right)+1}{f\left(\frac{1}{x}\right)-1}$.

Solution : Given $f(x)=\dfrac{3x+1}{3x-1}$

$\therefore \quad f\left(\dfrac{1}{x}\right)=\dfrac{3\cdot\frac{1}{x}+1}{3\cdot\frac{1}{x}-1}=\dfrac{\frac{3}{x}+1}{\frac{3}{x}-1} \;=\; \dfrac{-3+x}{3-x}$ [multiplying the numerator and the denomination by x]

or, $\dfrac{f\left(\frac{1}{x}\right)+1}{f\left(\frac{1}{x}\right)-1}=\dfrac{-3+x+3-x}{3+x-3+x}$, [By componendo –Dividendo]

$= \dfrac{6}{2x}=\dfrac{3}{x}$

Require value is $\dfrac{3}{x}$

Example 24. If $f(y)=\dfrac{y^3-3y^2+1}{y(1-y)}$, show that $f\left(\dfrac{1}{y}\right)=f(1-y)$

Solution : Given, $f(y)=\dfrac{y^3-3y^2+1}{y(1-y)}$

$\therefore \; f\left(\dfrac{1}{y}\right)=\dfrac{\left(\frac{1}{y}\right)^3-3\left(\frac{1}{y}\right)^2+1}{\frac{1}{y}\left(1-\frac{1}{y}\right)}=\dfrac{\frac{1-3y+y^3}{y^3}}{\frac{y-1}{y^2}}$

$= \dfrac{1-3y+y^3}{y^3}\times\dfrac{y^2}{y-1}=\dfrac{1-3y+y^3}{y(y-1)}$

again, $f(1-y)=\dfrac{(1-y)^3-3(1-y)^2+1}{(1-y)\{1-(1-y)\}}$

$= \dfrac{1-3y+3y^2-y^3-3(1-2y+y^2)+1}{(1-y)(1-1+y)}$

$$= \frac{1-3y+3y^2-y^3-3+6y-3y^2+1}{y(1-y)}$$

$$= \frac{-1+3y-y^3}{y(1-y)} = \frac{-(1-3y+y^3)}{-y(y-1)}$$

$$= \frac{1-3y+y^3}{y(y-1)}$$

$\therefore \; f\left(\frac{1}{y}\right) = f(1-y)$.

Exercise 2·2

1. Which one is the set of factors of 4 ?

   (a) {8,16,24...} (b) (1,2,3,4,8} (c) {2, 6, 8} (d) {1, 2}

2. If a relation of set B from set C is R, which one of the following is right ?

   (a) $R \subset C$ (b) $R \subset B$ (c) $R \subseteq C \times B$ (d) $C \times B \subseteq R$

3. If $A = \{6, 7, 8, 9, 10, 11, 12, 13\}$, answer the following questions :

   (i) Which one is builder method of set A ?

   (a) $\{x \in N: \; 6 < x < 13\}$ (b) $\{x \in N: \; 6 \le x < 13\}$

   (c) $\{x \in N: \; 6 \le x \le 13\}$ (d) $\{x \in N: \; 6 < x \le 13\}$

   (ii) Which one is the set of prime numbers ?

   (a) $\{6, 8, 10, 12\}$ (b) $\{7, \; 9, 11, 13\}$ (c) $\{7, \; 11, \; 13\}$ (d) $A = \{9, \; 12\}$

   (iii) Which is the set of multiple of 3 ?

   (a) $\{6, \; 9\}$ (b) $\{6, \; 11\}$ (c) $\{9, \; 12\}$ (d) $\{6, \; 9, \; 12\}$

   (iv) Which is the set of factor of greater even number ?

   (a) $\{1, \; 13\}$ (b) $\{1, \; 2, \; 3, \; 6\}$ (c) $\{1, \; 3, \; 9\}$ (d) $\{1, \; 2, \; 3, \; 4, \; 6, \; 12\}$

4. If $A = \{3, 4\}$, $B = \{2, 4\}$, find the relation between elements of A and B considering $x > y$.

5. If $C = \{2, 5\}$, $D = \{4, 6\}$ and find the relation between element of C and D considering relation $x + 1 < y$.

6. If $f(x) = x^4 + 5x - 3$, the find value of $f(-1)$, $f(2)$ and $f\left(\frac{1}{2}\right)$.

7. If $f(y) = y^3 + ky^3 - 4y - 8$, for which value of k will be $f(-2) = 0$.

8. If $f(x) = x^3 - 6x^2 + 11x - 6$, for which value of x will be $f(x) = 0$.

9. If $f(x)=\dfrac{2x+1}{2x-1}$, find the value of $\dfrac{f\left(\dfrac{1}{x^2}\right)+1}{f\left(\dfrac{1}{x^2}\right)-1}$.

10. If $g(x)=\dfrac{1+x^2+x^4}{x^2}$, show that, $g\left(\dfrac{1}{x^2}\right)=g(x^2)$

11. Find the domain and range of the following relations :
(a) $R=\{(2,1),(2,2),(2,3)\}$ (b) $S=\{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$
(c) $F=\left\{\left(\dfrac{1}{2},0\right),(1,1),(1,-1),\left(\dfrac{5}{2},2\right),\left(\dfrac{5}{2},-2\right)\right\}$

12. Express the relations in tabular method and find domain and range of following relations :
(a) $R=\{(x,y): x\in A,\ y\in A$ and $x+y=1\}$ where $A=\{-2,-1,0,1,2\}$
(b) $F=\{(x,y): x\in C,\ y\in C$ and $x=2y\}$ where $C=\{-1,0,1,1,3\}$

13. Plot the points $(-3,2)$, $(0,-5)$, $\left(\dfrac{1}{2},-\dfrac{5}{6}\right)$ on graph paper.

14. Plot the three points $(1,2)$, $(-1,1)$, $(11,7)$ on graph paper and show the three points are on the same straight line.

15. Universal set $U=\{x: x\in N$ and x is an odd number$\}$
$A=\{x\in N: 2\le x\le 7\}$
$B=\{x\in N: 3<x<6\}$
$C=\{x\in N: x^2>5$ and $x^3<130\}$
(a) Express A in tabular method.
(b) Find A' and $C-B$.
(c) Find $B\times C$ and $P(A\cap C)$.

Chapter Three

Algebraic Expressions

Algebraic formulae are used to solve many algebraic problems. Moreover, many algebraic expressions are presented by resolving them into factors. That is why the problem solved by algebraic formulae and the contents of resolving expressions into factors by making suitable for the students have been presented in this chapter. Moreover, different types of mathematical problems can be solved by resolving into factors with the help of algebraic formulae. In the previous class, algebraic formulae and their related corollaries have been discussed elaborately. In this chapter, those are reiterated and some of their applications are presented through examples. Besides, extension of the formulae of square and cube, resolution into factors using remainder theorem and formation of algebraic formulae and their applications in solving practical problems have been discussed here in detail.

At the end of the chapter, the students will be able to –

- Expand the formulae of square and cube by applying algebraic formulae
- Explain the remainder theorem and resolve into factors by applying the theorem
- Form algebraic formulae for solving real life problems and solve the problems by applying the formulae.

3·1 Algebraic Expressions

Meaningful organization of operational signs and numerical letter symbols is called algebraic expression. Such as, $2a+3b-4c$ is an algebraic expression. In algebraic expression, different types of information are expressed through the letters $a, b, c, p, q, r, m, n, x, y, z, ...$ etc. These alphabet are used to solve different types of problems related to algebraic expressions. In arithmetic, only positive numbers are used, where as, in algebra, both positive and negative numbers including zero are used. Algebra is the generalization form of arithmetic. The numbers used in algebraic expressions are constants, their values are fixed.

The letter symbols used in algebraic expressions are variables, their values are not fixed, they can be of any value.

3·2 Algebraic Formulae

Any general rule or resolution expressed by algebraic symbols is called Algebraic Formula. In class VII and VIII, algebraic formulae and related corollaries have been discussed. In this chapter, some applications are presented on the basis of that discussion.

Formula 1. $(a+b)^2 = a^2 + 2ab + b^2$

Formula 2. $(a-b)^2 = a^2 - 2ab + b^2$

Remark : It is seen from formula 1 and formula 2 that, adding $2ab$ or $-2ab$ with $a^2 + b^2$, we get a perfect square, i.e. we get, $(a+b)^2$ or $(a-b)^2$. Substituting $-b$ instead of b in formula 1, we get formula 2 :

$$\{a + (-b)\}^2 = a^2 + 2a(-b) + (-b)^2$$

That is, $(a-b)^2 = a^2 - 2ab + b^2$.

Corollary 1. $a^2 + b^2 = (a+b)^2 - 2ab$

Corollary 2. $a^2 + b^2 = (a-b)^2 + 2ab$

Corollary 3. $(a+b)^2 = (a-b)^2 + 4ab$

Proof : $(a+b)^2 = a^2 + 2ab + b^2$

$= a^2 - 2ab + b^2 + 4ab$

$= (a-b)^2 + 4ab$

Corollary 4. $(a-b)^2 = (a+b)^2 - 4ab$

Proof : $(a-b)^2 = a^2 - 2ab + b^2$

$= a^2 + 2ab + b^2 - 4ab$

$= (a+b)^2 - 4ab$

Corollary 5. $a^2 + b^2 = \dfrac{(a+b)^2 + (a-b)^2}{2}$

Proof : From formula 1 and formula 2,

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Adding, $2a^2 + 2b^2 = (a+b)^2 + (a-b)^2$

or, $2(a^2 + b^2) = (a+b)^2 + (a-b)^2$

Hence, $(a^2 + b^2) = \dfrac{(a+b)^2 + (a-b)^2}{2}$

Corollary 6. $ab = \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2$

Proof : From formula 1 and formula 2,

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Subtracting, $4ab = (a+b)^2 - (a-b)^2$

or, $ab = \frac{(a+b)^2}{4} - \frac{(a-b)^2}{4}$

Hence, $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

Remark : Product of any two quantities can be expressed as the difference of two squares by applying the corollary 6.

Formula 3. $a^2 - b^2 = (a+b)(a-b)$

That is, the difference of the squares of two expressions = sum of two expressions × difference of two expressions.

Formula 4. $(x+a)(x+b) = x^2 + (a+b)x + ab$

That is, $(x+a)(x+b) = x^2 +$ (algebraic sum of a and b) $x +$ (the product of a and b)

Extension of Formula for Square

There are three terms in the expression $a+b+c$. It can be considered the sum of two terms $(a+b)$ and c.

Therefore, by applying formula 1, the square of the expression $a+b+c$ is,

$$(a+b+c)^2 = \{(a+b)+c\}^2$$
$$= (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$
$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac.$$

Formula 5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$.

Corollary 7. $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac)$

Corollary 8. $2(ab+bc+ac) = (a+b+c)^2 - (a^2+b^2+c^2)$

Observe : Applying formula 5, we get,

(i) $(a+b-c)^2 = \{a+b+(-c)\}^2$

$$= a^2 + b^2 + (-c)^2 + 2ab + 2b(-c) + 2a(-c)$$
$$= a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$$

(ii) $(a-b+c)^2 = \{a+(-b)+c\}^2$

$$= a^2 + (-b)^2 + c^2 + 2a(-b) + 2(-b)c + 2ac$$
$$= a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$$

(iii) $(a-b-c)^2 = \{a+(-b)+(-c)\}^2$

$$= a^2 + (-b)^2 + (-c)^2 + 2a(-b) + 2(-b)(-c) + 2a(-c)$$

$= a^2 + b^2 + c^2 - 2ab + 2bc - 2ac$

Example 1. What is the square of $(4x + 5y)$?

Solution : $(4x + 5y)^2 = (4x)^2 + 2 \times (4x) \times (5y) + (5y)^2$

$= 16x^2 + 40xy + 25y^2$

Example 2. What is the square of $(3a - 7b)$?

Solution : $(3a - 7b)^2 = (3a)^2 - 2 \times (3a) \times (7b) + (7b)^2$

$= 9a^2 - 42ab + 49b^2$

Example 3. Find the square of 996 by applying the formula of square.

Solution : $(996)^2 = (1000 - 4)^2$

$= (1000)^2 - 2 \times 1000 \times 4 + (4)^2$

$= 1000000 - 8000 + 16$

$= 1000016 - 8000$

$= 992016$

Example 4. What is the square of $a + b + c + d$?

Solution : $(a + b + c + d)^2 = \{(a + b) + (c + d)\}^2$

$= (a + b)^2 + 2(a + b)(c + d) + (c + d)^2$

$= a^2 + 2ab + b^2 + 2(ac + ad + bc + bd) + c^2 + 2cd + d^2$

$= a^2 + 2ab + b^2 + 2ac + 2ad + 2bc + 2bd + c^2 + 2cd + d^2$

$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$

Activity : Find the square with the help of the formulae :
1. $3xy + 2ax$ 2. $4x - 3y$ 3. $x - 5y + 2z$

Example 5. Simplify :

$(5x + 7y + 3z)^2 + 2(7x - 7y - 3z)(5x + 7y + 3z) + (7x - 7y - 3z)^2$

Solution : Let, $5x + 7y + 3z = a$ and $7x - 7y - 3z = b$

$\therefore$ Given expression = $a^2 + 2.b.a + b^2$

$= a^2 + 2ab + b^2$

$= (a + b)^2$

$= \{(5x + 7y + 3z) + (7x - 7y - 3z)\}^2$

[substituting the values of a and b]

$= (5x + 7y + 3z + 7x - 7y - 3z)^2$

$= (12x)^2$

$= 144x^2$

Example 6. If $x-y=2$ and $xy=24$, what is the value of $x+y$?

Solution : $(x+y)^2=(x-y)^2+4xy=(2)^2+4\times 24=4+96=100$

$\therefore\ x+y=\pm\sqrt{100}=\pm 10$

Example 7. If $a^4+a^2b^2+b^4=3$ and $a^2+ab+b^2=3$, what is the value of a^2+b^2?

Solution : $a^4+a^2b^2+b^4=(a^2)^2+2a^2b^2+(b^2)^2-a^2b^2$

$=(a^2+b^2)^2-(ab)^2$

$=(a^2+b^2+ab)(a^2+b^2-ab)$

$=(a^2+ab+b^2)(a^2-ab+b^2)$

$\therefore\ 3=3(a^2-ab+b^2)$ [substituting the values]

or, $a^2-ab+b^2=\frac{3}{3}=1$

Now adding, $a^2+ab+b^2=3$ and $a^2-ab+b^2=1$ we get, $2(a^2+b^2)=4$

or, $a^2+b^2=\frac{4}{2}=2$

$\therefore\ a^2+b^2=2$

Example 8. Prove that, $(a+b)^4-(a-b)^4=8ab(a^2+b^2)$

Solution : $(a+b)^4-(a-b)^4=\{(a+b)^2\}^2-\{(a-b)^2\}^2$

$=\{(a+b)^2+(a-b)^2\}\{(a+b)^2-(a-b)^2\}$

$=2(a^2+b^2)\times 4ab$

[$\because (a+b)^2+(a-b)^2=2(a^2+b^2)$ and $(a+b)^2-(a-b)^2=4ab$]

$=8ab(a^2+b^2)$

$\therefore\ (a+b)^4-(a-b)^4=8ab(a^2+b^2)$

Example 9. If $a+b+c=15$ and $a^2+b^2+c^2=83$, what is the value of $ab+bc+ac$?

Soluton :

Here, $2(ab+bc+ac)$

$=(a+b+c)^2-(a^2+b^2+c^2)$

$=(15)^2-83$

$=225-83$

$=142$

$\therefore\ ab+bc+ac=\frac{142}{2}=71$

Alternative method,

We know ,

$(a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ac)$

or, $(15)^2=83+2(ab+bc+ac)$

or, $225-83=2(ab+bc+ac)$

or, $2(ab+bc+ac)=142$

$\therefore\ ab+bc+ac=\frac{142}{2}=71$

Example 10. If $a+b+c=2$ and $ab+bc+ac=1$, what is the value of $(a+b)^2+(b+c)^2+(c+a)^2$?

Solution : $(a+b)^2+(b+c)^2+(c+a)^2$

$= a^2+2ab+b^2+b^2+2bc+c^2+c^2+2ca+a^2$

$= (a^2+b^2+c^2+2ab+2bc+2ca)+(a^2+b^2+c^2)$

$= (a+b+c)^2+\{(a+b+c)^2-2(ab+bc+ac)\}$

$= (2)^2+(2)^2-2\times 1$

$= 4+4-2=8-2=6$

Example 11. Express $(2x+3y)(4x-5y)$ as the difference of two squares.

Solution : Let, $2x+3y=a$ and $4x-5y=b$

$\therefore$ Given expression = $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$

$=\left(\frac{2x+3y+4x-5y}{2}\right)^2-\left(\frac{2x+3y-4x+5y}{2}\right)^2$ [substituting the values of a and b]

$=\left(\frac{6x-2y}{2}\right)^2-\left(\frac{8y-2x}{2}\right)^2$

$=\left(\frac{2(3x-y)}{2}\right)^2-\left(\frac{2(4y-x)}{2}\right)^2$

$=(3x-y)^2-(4y-x)^2$

$\therefore\ (2x+3y)(4x-5y)=(3x-y)^2-(4y-x)^2$

Activity : 1. Simplify : $(4x+3y)^2+2(4x+3y)(4x-3y)+(4x-3y)^2$

2. If $x+y+z=12$ and $x^2+y^2+z^2=50$, find the value of $(x-y)^2+(y-z)^2+(z-x)^2$.

Exercise 3·1

1. Find the square with the help of the formulae :

(a) $2a+3b$ (b) $2ab+3bc$ (c) $x^2+\frac{2}{y^2}$ (d) $a+\frac{1}{a}$ (e) $4y-5x$ (f) $ab-c$

(g) $5x^2-y$ (h) $x+2y+4z$ (i) $3p+4q-5r$ (j) $3b-5c-2a$ (k) $ax-by-cz$

(l) $a-b+c-d$ (m) $2a+3x-2y-5z$ (n) 101 (o) 997 (p) 1007

2. Simplify :

(a) $(2a+7)^2+2(2a+7)(2a-7)+(2a-7)^2$

(b) $(3x+2y)^2+2(3x+2y)(3x-2y)+(3x-2y)^2$

(c) $(7p+3r-5x)^2-2(7p+3r-5x)(8p-4r-5x)+(8p-4r-5x)^2$

(d) $(2m+3n-p)^2+(2m-3n+p)^2-2(2m+3n-p)(2m-3n+p)$

(e) $6\cdot35\times6\cdot35+2\times6\cdot35\times3\cdot65+3\cdot65\times3\cdot65$

(f) $5874\times5874+3774\times3774-7548\times5874$

(g) $\dfrac{7529\times7529-7519\times7519}{7529+7519}$

(h) $\dfrac{2345\times2345-759\times759}{2345-759}$

3. If $a-b=4$ and $ab=60$, what is the value of $a+b$?
4. If $a+b=7$ and $ab=12$, what is the value of $a-b$?
5. If $a+b=9m$ and $ab=18m^2$, what is the value of $a-b$?
6. If $x-y=2$ and $xy=63$, what is the value of x^2+y^2 ?
7. If $x-\dfrac{1}{x}=4$, prove that, $x^4+\dfrac{1}{x^4}=322$.
8. If $2x+\dfrac{2}{x}=3$, what is the value of $x^2+\dfrac{1}{x^2}$?
9. If $a+\dfrac{1}{a}=2$, show that, $a^2+\dfrac{1}{a^2}=a^4+\dfrac{1}{a^4}$.
10. If $a+b=\sqrt{7}$ and $a-b=\sqrt{5}$, prove that, $8ab(a^2+b^2)=24$.
11. If $a+b+c=9$ and $ab+bc+ca=31$, find the value of $a^2+b^2+c^2$.
12. If $a^2+b^2+c^2=9$ and $ab+bc+ca=8$, what is the value of $(a+b+c)^2$?
13. If $a+b+c=6$ and $a^2+b^2+c^2=14$, find the value of $(a-b)^2+(b-c)^2+(c-a)^2$.
14. If $x+y+z=10$ and $xy+yz+zx=31$, what is the value of $(x+y)^2+(y+z)^2+(z+x)^2$?
15. If $x=3, y=4$ and $z=5$ find the value of $9x^2+16y^2+4z^2-24xy-16yz+12zx$.
16. Prove that, $\left\{\left(\dfrac{x+y}{2}\right)^2-\left(\dfrac{x-y}{2}\right)^2\right\}^2=\left(\dfrac{x^2+y^2}{2}\right)^2-\left(\dfrac{x^2-y^2}{2}\right)^2$.
17. Express $(a+2b)(3a+2c)$ as the difference of two squares.
18. Express $(x+7)(x-9)$ as the difference of two squares.
19. Express $x^2+10x+24$ as the difference of two squares.

20. If $a^4 + a^2b^2 + b^4 = 8$ and $a^2 + ab + b^2 = 4$, find the value of (i) $a^2 + b^2$, (ii) ab.

3·3 Formulae of Cubes

Formula 6. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$= a^3 + b^3 + 3ab(a+b)$$

Proof : $(a+b)^3 = (a+b)(a+b)^2$

$$= (a+b)(a^2 + 2ab + b^2)$$
$$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + b^3 + 3ab(a+b)$$

Corollary 9. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

Formula 7. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$= a^3 - b^3 - 3ab(a-b)$$

Proof : $(a-b)^3 = (a-b)(a-b)^2$

$$= (a-b)(a^2 - 2ab + b^2)$$
$$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$$
$$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3$$
$$= a^3 - b^3 - 3ab(a-b)$$

Observe : Substituting $-b$ instead of b in formula 6, we get formula 7 :

$$\{a + (-b)\}^3 = a^3 + (-b)^3 + 3a(-b)\{a + (-b)\}$$

That is, $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

Corollary 10. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

Formula 8. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Proof : $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$= (a+b)\{(a+b)^2 - 3ab\}$$
$$= (a+b)(a^2 + 2ab + b^2 - 3ab)$$
$$= (a+b)(a^2 - ab + b^2)$$

Formula 9. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Proof : $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

$$= (a-b)\{(a-b)^2 + 3ab\}$$

$= (a-b)(a^2 - 2ab + b^2 + 3ab)$

$= (a-b)(a^2 + ab + b^2)$

Example 12. Find the cube of $2x + 3y$.

Solution : $(2x+3y)^3 = (2x)^3 + 3(2x)^2 \cdot 3y + 3 \cdot 2x(3y)^2 + (3y)^3$

$= 8x^3 + 3 \cdot 4x^2 \cdot 3y + 3 \cdot 2x \cdot 9y^2 + 27y^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

Example 13. Find the cube of $2x - y$.

Solution : $(2x-y)^3 = (2x)^3 - 3 \cdot (2x)^2 y + 3 \cdot 2x \cdot y^2 - y^3$

$= 8x^3 - 3 \cdot 4x^2 y + 6xy^2 - y^3$

$= 8x^3 - 12x^2 y + 6xy^2 - y^3$

Activity : Find the cube with the help of the formulae.
1. $3x + 2y$ 2. $3x - 4y$ 3. 397

Example 14. If $x = 37$, what is the value of $8x^3 + 72x^2 + 216x + 216$?

Solution : $8x^3 + 72x^2 + 216x + 216$

$= (2x)^3 + 3 \cdot (2x)^2 \cdot 6 + 3 \cdot 2x.(6)^2 + (6)^3$

$= (2x+6)^3$

$= (2 \times 37 + 6)^3$ [substituting the values]

$= (74+6)^3$

$= (80)^3$

$= 512000$

Example 15. If $x - y = 8$ and $xy = 5$, what is the value of $x^3 - y^3 + 8(x+y)^2$?

Solution : $x^3 - y^3 + 8(x+y)^2$

$= (x-y)^3 + 3xy(x-y) + 8\{(x-y)^2 + 4xy\}$

$= (8)^3 + 3 \times 5 \times 8 + 8(8^2 + 4 \times 5)$ [substituting the values]

$= 8^3 + 15 \times 8 + 8(64 + 20)$

$= 8^3 + 15 \times 8 + 8 \times 84$

$= 8(8^2 + 15 + 84)$

$= 8(64 + 15 + 84)$

$= 8 \times 163$

$= 1304$

Example 16. If $a^2-\sqrt{3}a+1=0$, what is the value of $a^3+\frac{1}{a^3}$?

Solution : Given that, $a^2-\sqrt{3}a+1=0$

or, $a^2+1=\sqrt{3}a$ or, $\frac{a^2+1}{a}=\sqrt{3}$

or, $\frac{a^2}{a}+\frac{1}{a}=\sqrt{3}$ or, $a+\frac{1}{a}=\sqrt{3}$

$\therefore$ Given expression $= a^3+\frac{1}{a^3}$

$=\left(a+\frac{1}{a}\right)^3-3a\cdot\frac{1}{a}\left(a+\frac{1}{a}\right)$

$=\left(\sqrt{3}\right)^3-3\left(\sqrt{3}\right)$ $[\because a+\frac{1}{a}=\sqrt{3}]$

$=3\sqrt{3}-3\sqrt{3}$

$=0$

Example 17. Simplify :

$(a-b)(a^2+ab+b^2)+(b-c)(b^2+bc+c^2)+(c-a)(c^2+ca+a^2)$

Solution : $(a-b)(a^2+ab+b^2)+(b-c)(b^2+bc+c^2)+(c-a)(c^2+ca+a^2)$

$=a^3-b^3+b^3-c^3+c^3-a^3$

$=0$

Example 18. If $a=\sqrt{3}+\sqrt{2}$, prove that, $a^3+\frac{1}{a^3}=18\sqrt{3}.$

Solution : Given that, $a=\sqrt{3}+\sqrt{2}$

$\therefore\ \frac{1}{a}=\frac{1}{\sqrt{3}+\sqrt{2}}$

$=\frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}$ [multiplying numerator and denominator by $\left(\sqrt{3}-\sqrt{2}\right)$]

$=\frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$ $=\frac{\sqrt{3}-\sqrt{2}}{3-2}$

$=\sqrt{3}-\sqrt{2}$

$\therefore\ a+\frac{1}{a}=\left(\sqrt{3}+\sqrt{2}\right)+\left(\sqrt{3}-\sqrt{2}\right)$

$=\sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2}=2\sqrt{3}$

Now, $a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\cdot a\cdot\frac{1}{a}\left(a+\frac{1}{a}\right)$

$= \left(2\sqrt{3}\right)^3-3\left(2\sqrt{3}\right)$ $[\because a+\frac{1}{a}=2\sqrt{3}]$

$= 2^3\cdot\left(\sqrt{3}\right)^3-3\times2\sqrt{3}$

$= 8\cdot3\sqrt{3}-6\sqrt{3}$

$= 24\sqrt{3}-6\sqrt{3}$

$= 18\sqrt{3}$ (proved)

Activity : 1. If $x=-2$, what is the value of $27x^3-54x^2+36x-8$?
2. If $a+b=5$ and $ab=6$, find the value of $a^3+b^3+4(a-b)^2$.
3. If $x=\sqrt{5}+\sqrt{3}$, find the value of $x^3+\frac{1}{x^3}$.

Exercise 3·2

1. Find the cube with the help of the formulae :
(a) $2x+5$ (b) $2x^2+3y^2$ (c) $4a-5x^2$ (d) $7m^2-2n$ (e) 403 (f) 998
(g) $2a-b-3c$ (h) $2x+3y+z$
2. Simplify :
(a) $(4a-3b)^3-3(4a-3b)^2(2a-3b)+3(4a-3b)(2a-3b)^2-(2a-3b)^3$
(b) $(2x+y)^3+3(2x+y)^2(2x-y)+3(2x+y)(2x-y)^2+(2x-y)^3$
(c) $(7x+3b)^3-(5x+3b)^3-6x(7x+3b)(5x+3b)$
(d) $(x-15)^3+(16-x)^3+3(x-15)(16-x)$
(e) $(a+b+c)^3-(a-b-c)^3-6(b+c)\{a^2-(b+c)^2\}$
(f) $(m+n)^6-(m-n)^6-12mn(m^2-n^2)^2$
(g) $(x+y)(x^2-xy+y^2)+(y+z)(y^2-yz+z^2)+(z+x)(z^2-zx+x^2)$
(h) $(2x+3y-4z)^3+(2x-3y+4z)^3+12x\{4x^2-(3y-4z)^2\}$
3. If $a-b=5$ and $ab=36$, what is the value of a^3-b^3 ?
4. If $a^3-b^3=513$ and $a-b=3$, what is the value of ab ?
5. If $x=19$ and $y=-12$, find the value of $8x^3+36x^2y+54xy^2+27y^3$.
6. If $a=15$, what is the value of $8a^3+60a^2+150a+130$?
7. If $a=7$ and $b=-5$, what is the value of

$(3a-5b)^3+(4b-2a)^3+3(a-b)(3a-5b)(4b-2a)$?

8. If $a+b=m, a^2+b^2=n$ and $a^3+b^3=p^3$, show that, $m^3+2p^3=3mn$.
9. If` $x+y=1$, show that, $x^3+y^3-xy=(x-y)^2$
10. If $a+b=3$ and $ab=2$, find the value of (a) a^2-ab+b^2 and (b) a^3+b^3.
11. If $a-b=5$ and $ab=36$, find the value of (a) a^2+ab+b^2 and (b) a^3-b^3.
12. If $m+\frac{1}{m}=a$, find the value of $m^3+\frac{1}{m^3}$.
13. If $x-\frac{1}{x}=p$, find the value of $x^3-\frac{1}{x^3}$.
14. If $a-\frac{1}{a}=1$, show that, $a^3-\frac{1}{a^3}=4$.
15. If $a+b+c=0$, show that,

    (a) $a^3+b^3+c^3=3abc$ (b) $\frac{(b+c)^2}{3bc}+\frac{(c+a)^2}{3ca}+\frac{(a+b)^2}{3ab}=1$.
16. If $p-q=r$, show that, $p^3-q^3-r^3=3pqr$
17. If $2x-\frac{2}{x}=3$, show that, $8\left(x^3-\frac{1}{x^3}\right)=63$.
18. If $a=\sqrt{6}+\sqrt{5}$, find the value of $\frac{a^6-1}{a^3}$.
19. If $x^3+\frac{1}{x^3}=18\sqrt{3}$, prove that, $x=\sqrt{3}+\sqrt{2}$.
20. If $a^4-a^2+1=0$, prove that, $a^3+\frac{1}{a^3}=0$.

3·4 Resolution into Factors

If an expression is equal to the product of two or more expressions, each of the latter expressions is called a factor of the former expression.

After finding the possible factors of any algebraic expression and then expressing the expression as the product of these factors are called factorization or resolution into factors.

The algebraic expressions may consist of one or more terms. So, the factors may also contain one or more terms.

Some process of resolving expressions into factors :

(a) If any polynomial expression has common factor in every term, at first they are to be found out. For example,

(*i*) $3a^2b+6ab^2+12a^2b^2=3ab(a+2b+4ab)$

$(ii)\ 2ab(x-y)+2bc(x-y)+3ca(x-y)=(x-y)(2ab+2bc+3ca)$

(b) Expressing an expression in the form of a perfect square ;

Example 1. Resolve into factors : $4x^2+12x+9$.

Solution : $4x^2+12x+9=(2x)^2+2\times 2x\times 3+(3)^2$

$= (2x+3)^2=(2x+3)(2x+3)$

Example 2. Resolve into factors : $9x^2-30xy+25y^2$.

Solution : $9x^2-30xy+25y^2$

$= (3x)^2-2\times 3x\times 5y+(5y)^2$

$= (3x-5y)^2=(3x-5y)(3x-5y)$

(c) Expressing an expression as the difference of two squares and then applying the formula $a^2-b^2=(a+b)(a-b)$:

Example 3. Reslove into factors : $a^2-1+2b-b^2$.

Solution : $a^2-1+2b-b^2=a^2-(b^2-2b+1)$

$= a^2-(b-1)^2=\{a+(b-1)\}\{a-(b-1)\}$

$= (a+b-1)(a-b+1)$

Example 4. Resolve into factors : a^4+64b^4.

Solution : $a^4+64b^4=(a^2)^2+(8b^2)^2$

$= (a^2)^2+2\times a^2\times 8b^2+(8b^2)^2-16a^2b^2$

$= (a^2+8b^2)^2-(4ab)^2$

$= (a^2+8b^2+4ab)(a^2+8b^2-4ab)$

$= (a^2+4ab+8b^2)(a^2-4ab+8b^2)$

Activity : Resolve into factors :

1. $abx^2+acx^3+adx^4$ 2. $xa^2-144xb^2$ 3. $x^2-2xy-4y-4$

(d) Using the formula $x^2+(a+b)x+ab=(x+a)(x+b)$:

Example 5. Resolve into factors : $x^2+12x+35$.

Solution : $x^2+12x+35=x^2+(5+7)x+5\times 7$

$= (x+5)(x+7)$

In this method, a polynomial of the form x^2+px+q can be factorized, if two integers a and b can be found so that, it is $a+b=p$ and $ab=q$. For this, two factors of q with their signs are to be taken whose algebraic sum is p. If $q>0$, a and b will be of same signs and if $q<0$, a and b will be of opposite signs.

Example 6. Resolve into factors : $x^2 - 5x + 6$.

Solution : $x^2 - 5x + 6 = x^2 + (-2-3)x + (-2)(-3)$

$= (x-2)(x-3)$

Example 7. Resolve into factors : $x^2 - 2x - 35$.

Solution : $x^2 - 2x - 35$

$= x^2 + (-7+5)x + (-7)(+5)$

$= (x-7)(x+5)$

Example 8. Resolve into factors : $x^2 + x - 20$.

Solution : $x^2 + x - 20$

$= x^2 + (5-4)x + (5)(-4)$

$= (x+5)(x-4)$

(e) By middle term break-up method of polynomial of the form of $ax^2 + bx + c$:

If $ax^2 + bx + c = (rx + p)(sx + q)$

$ax^2 + bx + c = rsx^2 + (rq + sp)x + pq$

That is, $a = rs, b = rq + sp$ and $c = pq$.

Hence, $ac = rspq = (rq)(sp)$ and $b = rq + sp$

Therefore, to determine factors of the polynomial $ax^2 + bx + c$, ac that is, the product of the coefficient of x^2 and the term free from x are to be expressed into two such factors whose algebraic sum is equal to b, the coefficient of x.

Example 9. Resolve into factors : $12x^2 + 35x + 18$.

Solution : $12x^2 + 35x + 18$

Here, $12 \times 18 = 216 = 27 \times 8$ and $27 + 8 = 35$

$\therefore\ 12x^2 + 35x + 18 = 12x^2 + 27x + 8x + 18$

$= 3x(4x+9) + 2(4x+9)$

$= (4x+9)(3x+2)$

Example 10. Resolve into factors : $3x^2 - x - 14$.

Solution : $3x^2 - x - 14 = 3x^2 - 7x + 6x - 14$

$= x(3x-7) + 2(3x-7)$

$= (3x-7)(x+2)$

Activity : Resolve into factors :		
1. $x^2 + x - 56$	2. $16x^3 - 46x^2 + 15x$	3. $12x^2 + 17x + 6$

(f) Expressing the expression in the form of perfect cubes :

Example 11. Resolve into factors : $8x^3+36x^2y+54xy^2+27y^3$.

Solution : $8x^3+36x^2y+54xy^2+27y^3$

$= (2x)^3+3\times(2x)^2\times 3y+3\times 2x\times(3y)^2+(3y)^3$

$= (2x+3y)^3=(2x+3y)(2x+3y)(2x+3y)$

(g) Applying the formulae : $a^3+b^3=(a+b)(a^2-ab+b^2)$ **and** $a^3-b^3=(a-b)(a^2+ab+b^2)$ **:**

Example 12. Resolve into factors : (*i*) $8a^3+27b^3$ (*ii*) a^6-64

Solution : (*i*) $8a^3+27b^3=(2a)^3+(3b)^3$

$= (2a+3b)\{(2a)^2-2a\times 3b+(3b)^2\}$

$= (2a+3b)(4a^2-6ab+9b^2)$

(*ii*) $a^6-64 = (a^2)^3-(4)^3$

$= (a^2-4)\{(a^2)^2+a^2\times 4+(4)^2\}$

$= (a^2-4)(a^4+4a^2+16)$

But, $a^2-4=a^2-2^2=(a+2)(a-2)$

and $a^4+4a^2+16=(a^2)^2+(4)^2+4a^2$

$= (a^2+4)^2-2(a^2)(4)+4a^2$

$= (a^2+4)^2-4a^2$

$= (a^2+4)^2-(2a)^2$

$= (a^2+4+2a)(a^2+4-2a)$

$= (a^2+2a+4)(a^2-2a+4)$

Alternative method :

$a^6-64 = (a^3)^2-8^2$

$= (a^3+8)(a^3-8)$

$= (a^3+2^3)(a^3-2^3)$

$= (a+2)(a^2-2a+4)\times(a-2)(a^2+2a+4)$

$= (a+2)(a-2)(a^2+2a+4)(a^2-2a+4)$

$\therefore\ a^6-64$

$=(a+2)(a-2)(a^2+2a+4)(a^2-2a+4)$

Activity : Resolve into factors:

1. $2x^4+16x$ 2. $8-a^3+3a^2b-3ab^2+b^3$ 3. $(a+b)^3+(a-b)^3$

(h) Factors of the expression with fractional coefficients :

Factors of the expressions with fraction may be expressed in different ways.

For example, $a^3+\dfrac{1}{27}=a^3+\dfrac{1}{3^3}=\left(a+\dfrac{1}{3}\right)\left(a^2-\dfrac{a}{3}+\dfrac{1}{9}\right)$

Again, $a^3+\dfrac{1}{27}=\dfrac{1}{27}(27a^3+1)=\dfrac{1}{27}\{(3a)^3+(1)^3\}$

$$= \frac{1}{27}(3a+1)(9a^2-3a+1)$$

Here, in the second solution, the factors involving the variables are with integral coefficients. This result can be expressed as the first solution :

$$\frac{1}{27}(3a+1)(9a^2-3a+1)$$
$$= \frac{1}{3}(3a+1)\times\frac{1}{9}(9a^2-3a+1)$$
$$= \left(a+\frac{1}{3}\right)\left(a^2-\frac{a}{3}+\frac{1}{9}\right)$$

Example 13. Resolve into factors : $x^3+6x^2y+11xy^2+6y^3$.

Solution : $x^3+6x^2y+11xy^2+6y^3$

$$= \{x^3+3\cdot x^2\cdot 2y+3\cdot x(2y)^2+(2y)^3\}-xy^2-2y^3$$
$$= (x+2y)^3-y^2(x+2y)$$
$$= (x+2y)\{(x+2y)^2-y^2\}$$
$$= (x+2y)(x+2y+y)(x+2y-y)$$
$$= (x+2y)(x+3y)(x+y)$$
$$= (x+y)(x+2y)(x+3y)$$

Activity : Resolve into factors:

1. $\frac{1}{2}x^2+\frac{7}{6}x+\frac{1}{3}$ 2. $a^3+\frac{1}{8}$ 3. $16x^2-25y^2-8xz+10yz$

Exercise 3·3

Resolve into factors **(1 to 43) :**

1. $a^2+ab+ac+bc$
2. $ab+a-b-1$
3. $(x-y)(x+y)+(x-y)(y+z)+(x-y)(z+x)$
4. $ab(x-y)-bc(x-y)$
5. $9x^2+24x+16$
6. a^4-27a^2+1
7. $x^4-6x^2y^2+y^4$
8. $(a^2-b^2)(x^2-y^2)+4abxy$
9. $4a^2-12ab+9b^2-4c^2$
10. $9x^4-45a^2x^2+36a^4$
11. $a^2+6a+8-y^2+2y$
12. $16x^2-25y^2-8xz+10yz$
13. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
14. $x^2+13x+36$
15. x^4+x^2-20
16. $a^2-30a+216$

17. $x^6y^6 - x^3y^3 - 6$
18. $a^8 - a^4 - 2$
19. $a^2b^2 - 8ab - 105$
20. $x^2 - 37a - 650$
21. $4x^4 - 25x^2 + 36$
22. $12x^2 - 38x + 20$
23. $9x^2y^2 - 5xy^2 - 14y^2$
24. $4x^4 - 27x^2 - 81$
25. $ax^2 + (a^2 + 1)x + a$
26. $3(a^2 + 2a)^2 - 22(a^2 + 2a) + 40$
27. $14(x + z)^2 - 29(x + z)(x + 1) - 15(x + 1)^2$
28. $(4a - 3b)^2 - 2(4a - 3b)(a + 2b) - 35(a + 2b)^2$
29. $(a - 1)x^2 + a^2xy + (a + 1)y^2$
30. $24x^4 - 3x$
31. $(a^2 + b^2)^3 + 8a^3b^3$
32. $x^3 + 3x^2 + 3x + 2$
33. $a^3 - 6a^2 + 12a - 9$
34. $a^3 - 9b^3 + (a + b)^3$
35. $8x^3 + 12x^2 + 6x - 63$
36. $8a^3 + \frac{b^3}{27}$
37. $a^3 - \frac{1}{8}$
38. $\frac{a^6}{27} - b^6$
39. $4a^2 + \frac{1}{4a^2} - 2 + 4a - \frac{1}{a}$
40. $(3a + 1)^3 - (2a = 3)^3$
41. $(x + 5)(x - 9) - 15$
42. $(x + 2)(x + 3)(x + 4)(x + 5) - 48$
43. $(x - 1)(x - 3)(x - 5)(x - 7) - 64$
44. Show that, $x^3 + 9x^2 + 26x + 24 = (x + 2)(x + 3)(x + 4)$
45. Show that, $(x + 1)(x + 2)(3x - 1)(3x - 4) = (3x^2 + 2x - 1)(3x^2 + 2x - 8)$

3·5 Remainder Theorem

We observe the following example :

If $6x^2 - 7x + 5$ is divided by $x - 1$, the what is quotient and remainder?

Dividing $6x^2 - 7x + 5$ by $x - 1$ in common way, we get,

$$\begin{array}{rl} x-1\ ) & 6x^2 - 7x + 5\ (\ 6x - 1 \\ & \underline{6x^2 \underset{+}{-} 6x} \\ & \quad -x + 5 \\ & \quad \underline{\underset{+}{-}x \underset{-}{+} 1} \\ & \qquad\quad 4 \end{array}$$

Here, $x - 1$ is divisor, $6x^2 - 7x + 5$ is dividend, $6x - 1$ is quotient and 4 is remainder.

We know, dividend = divisor × quotient + remainder

Now, if we indicate the dividend by $f(x)$, the quotient by $h(x)$, the remainder by r and the divisor by $(x-a)$, from the above formula, we get,

$f(x)=(x-a)\cdot h(x)+r$ .... (i) this formula is true to all values of a.

Substituting $x=a$ in both sides of (*i*), we get,

$f(a)=(a-a)\cdot h(a)+r=0\cdot h(a)+r=r$

Hence, $r=f(a)$.

Therefore, if $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. This formula is known as remainder theorem. That is, the remainder theorem gives the remainder when a polynomial $f(x)$ of positive degree is divided by $(x-a)$ without performing actual division. The degree of the divisor polynomial $(x-a)$ is 1. If the divisor is a factor of the dividend, the remainder will be zero and if it is not zero, the remainder will be a number other than zero.

Proposition : If the degree of $f(x)$ is positive and $a\neq 0$, $f(x)$ is divided by $(ax+b)$, remainder is $f\left(-\frac{b}{a}\right)$.

Proof : Degree of the divisor $ax+b$, $(a\neq 0)$ is 1.

Hence, we can write,

$$f(x)=(ax+b)\cdot h(x)+r=a\left(x+\frac{b}{a}\right)\cdot h(x)+r$$

$\therefore\ f(x)=\left(x+\frac{b}{a}\right)\cdot a\cdot h(x)+r$

Observe that, if $f(x)$ is divided by $\left(x+\frac{b}{a}\right)$, quotient is $a\cdot h(x)$ and remainder is r.

Here, divisor = $x-\left(-\frac{b}{a}\right)$

Hence, according to remainder theorem, $r=f\left(-\frac{b}{a}\right)$

Therefore, if $f(x)$ is divided by $(ax+b)$, remainder is $f\left(-\frac{b}{a}\right)$.

Corollary : $(x-a)$ will be a factor of $f(x)$, if and only if $f(a)=0$.

Proof : Let, $f(a)=0$

Therefore, according to remainder theorem, if $f(x)$ is divided by $(x-a)$, the remainder will be zero. That is, $(x-a)$ will be a factor of $f(x)$.

Conversely, let, $(x-a)$ is a factor of $f(x)$.

Therefore, $f(x)=(x-a)\cdot h(x)$, where $h(x)$ is a polynomial.

Putting $x=a$ in both sides, we get,

$f(a) = (a - a) \cdot h(a) = 0$

$\therefore\ f(a) = 0$.

Hence, any polynomial $f(x)$ will be divisible by $(x - a)$, if and only if $f(a) = 0$. This formula is known as factorisation theorem or factor theorem.

Corollary : If $a \neq 0$, the polynomial $ax + b$ will be a factor of any polynomial $f(x)$, if and only if $f\left(-\frac{b}{a}\right) = 0$.

Proof : $a \neq 0, ax + b = a\left(x + \frac{b}{a}\right)$ will be a factor of $f(x)$, if and only if $\left(x + \frac{b}{a}\right) = x - \left(-\frac{b}{a}\right)$ is a factor of $f(x)$, i.e. if and only if $f\left(-\frac{b}{a}\right) = 0$. This method of determining the factors of polynomial with the help of the remainder theorem is also called the Vanishing method.

Example 1. Resolve into factors : $x^3 - x - 6$.

Solution : Here, $f(x) = x^3 - x - 6$ is a polynomial. The factors of the constant -6 are $\pm 1, \pm 2, \pm 3$ and ± 6.

Putting, $x = 1, -1$, we see that the value of $f(x)$ is not zero.

But putting $x = 2$, we see that the value of $f(x)$ is zero.

i.e., $f(2) = 2^3 - 2 - 6 = 8 - 2 - 6 = 0$

Hence, $x - 2$ is a factor of $f(x)$

$$\begin{aligned}\therefore\ f(x) &= x^3 - x - 6\\ &= x^3 - 2x^2 + 2x^2 - 4x + 3x - 6\\ &= x^2(x-2) + 2x(x-2) + 3(x-2)\\ &= (x-2)(x^2 + 2x + 3)\end{aligned}$$

Example 2. Resolve into factors : $x^3 - 3xy^2 + 2y^3$.

Solution: Here, consider x a variable and y a constant.

We consider the given expression a polynomial of x.

Let, $f(x) = x^3 - 3xy^2 + 2y^3$

Then, $f(y) = y^3 - 3y \cdot y^2 + 2y^3 = 3y^3 - 3y^3 = 0$

$\therefore\ (x - y)$ is a factor of $f(x)$.

Now, $x^3 - 3xy^2 + 2y^3$

$= x^3 - x^2y + x^2y - xy^2 - 2xy^2 + 2y^3$

Again let, $g(x) = x^2 + xy - 2y^2$

$\therefore\ g(y) = y^2 + y^2 - 2y^2 = 0$

$= x^2(x-y)+xy(x-y)-2y^2(x-y)$
$= (x-y)(x^2+xy-2y^2)$
$= (x-y)(x^2+2xy-xy-2y^2)$
$= (x-y)\{x(x+2y)-y(x+2y)\}$
$= (x-y)(x+2y)(x-y)$
$= (x-y)^2(x+2y)$

$\therefore (x-y)$ is a factor of $g(x)$
$\therefore\ x^2+xy-2y^2$
$= x^2-xy+2xy-2y^2$
$= x(x-y)+2y(x-y)$
$= (x-y)(x+2y)$
$\therefore\ x^3-3xy^2+2y^3 = (x-y)^2(x+2y)$

Example 3. Resolve into factors : $54x^4+27x^3a-16x-8a$.

Solution : Let, $f(x)=54x^4+27x^3a-16x-8a$

then, $f\left(-\frac{1}{2}a\right)=54\left(-\frac{1}{2}a\right)^4+27a\left(-\frac{1}{2}a\right)^3-16\left(-\frac{1}{2}a\right)-8a$

$= \frac{27}{8}a^4-\frac{27}{8}a^4+8a-8a=0$

$\therefore\ x-\left(-\frac{1}{2}a\right)=x+\frac{a}{2}$ i.e., $2x+a$ is a factor of $f(x)$.

Now, $54x^4+27x^3a-16x-8a=27x^3(2x+a)-8(2x+a)=(2x+a)(27x^3-8)$

$=(2x+a)\{(3x)^3-(2)^3\}=(2x+a)(3x-2)(9x^2+6x+4)$

Activity : Resolve into factors :
1. $x^3-21x-20$ 2. $2x^3-3x^2+3x-1$ 3. $x^3+6x^2+11x+6$

Exercise 3·4

Resolve into factors :

1. $6x^2-7x+1$
2. $3a^3+2a+5$
3. $x^3-7xy^2-6y^3$
4. x^2-5x-6
5. $2x^2-x-3$
6. $3x^2-7x-6$
7. x^3+2x^2-5x-6
8. x^3+4x^2+x-6
9. $a^3+3a+36$
10. a^4-4a+3
11. $a^3-a^2-10a-8$
12. x^3-3x^2+4x-4
13. $a^3-7a^2b+7ab^2-b^3$
14. x^3-x-24
15. $x^3+6x^2y+11xy^2+6y^3$
16. $2x^4-3x^3-3x-2$
17. $4x^4+12x^3+7x^2-3x-2$
18. $x^6-x^5+x^4-x^3+x^2-x$
19. $4x^3-5x^2+5x-1$
20. $18x^3+15x^2-x-2$

3·6 Forming and applying algebraic formulae in solving real life problems

In our daily business we face the realistic problems in different time and in different ways. These problems are described linguistically. In this section, we shall discuss the formation of algebraic formulae and their applications in solving different problems of real surroundings which are described linguistically. As a result of this discussion the students on the one hand, will get the conception about the application of mathematics in real surroundings, on the other hand, they will be eager to learn mathematics for their understanding of the involvement of mathematics with their surroundings.

Methods of solving the problems :

(a) At first the problem will have to be observed carefully and to read attentively and then to identify which are unknown and which are to be determined.

(b) One of the unknown quantities is to be denoted with any variable (say x). Then realising the problem well, express other unknown quantities in terms of the same variable (x).

(c) The problem will have to be splitted into small parts and express them by algebraic expressions.

(d) Using the given conditions, the small parts together are to be expressed by an equation.

(e) The value of the unknown quantity x is to be found by solving the equation.

Different formulae are used in solving the problems based on real life. The formulae are mentioned below :

(1) Related to Payable or Attainable :

Payable or attainable, $A =$ Tk. qn

where, q = amount of money payable or attainable per person,

n = number of person.

(2) Related to Time and Work :

If some persons perform a work,

Amount of work done, $W = qnx$

where, q = portion of a work performed by every one in unit of time.

n = number of performing of work

x = total time of doing work

$W =$ portion of a work done by n persons in time x.

(3) Related to Time and Distance :

Distance at a definite time, $d = vt$.

where, v = speed per hour

t = total time.

(4) Related to pipe and water tank :

Amount of water in a tank at a definite time, $Q(t) = Q_o \pm qt$

where, Q_o = amount of stored water in a tank at the time of opening the pipe

q = amount of water flowing in or flowing out by the pipe in a unit time.

t = time taken.

$Q(t)$ = amount of water in the tank in time t (+sign, at the time o f flowing of water in and −sign, at the time of flowing of water out are to be used).

(5) Related to percentage :

$p = br$,

where, b = total quantity

r = (rate of) percentage by fraction $= \frac{s}{100} = s\%$

p = (rate of) percentage by parts = $s\%$ of b

(6) Related to profit and loss :

$S = C(I \pm r)$;

in case of profit, $S = C(I + r)$

in case of loss, $S = C(I - r)$

where, S (Tk.) = selling price

C (Tk.) = cost price

I = profit

r = rate of profit or loss

(7) Related to investment and profit :

In the case of simple profit,

$I = Pnr$ (taka)

$A = P + I = P + Pnr = P(1 + nr)$ (taka)

In the case of compound profit,

$A = P(1 + r)^n$

where, I = profit after time n

n = specific time

P = principal

r = profit of unit principal at unit time

A = principal with profit after time n.

Example 1. For a function of Annual Sports, members of an association made a budget of Tk. 45000 and divided that every member would subscribe equally. But 5 members refused to subscribe. As a result, amount of subscription of each member increased by Tk. 15 per head. How many members were in the association ?

Solution : Let the number of members of the association be x and amount of subscription per head be Tk. q Then total amount of subscription A = Tk. qx.

Actually numbers of members were $(x-5)$ and amount of subscription per head became Tk. $(q+15)$.

Then, total amount of subscription = Tk. $(x-5)(q+15)$

By the question, $qx = (x-5)(q+15)..........(i)$

and $qx = 45{,}000..........(ii)$

From equation (i), we get, $qx = (x-5)(q+15)$

or, $qx = qx - 5q + 15x - 75$

or, $5q = 15x - 75 = 5(3x - 15)$

$\therefore\ q = 3x - 15...........(iii)$

Putting the value of q in equation (ii),

$(3x-15)\times x = 45000$

or, $3x^2 - 15x = 45000$

or, $x^2 - 5x = 15000$ [dividing both sides by 3]

or, $x^2 - 5x - 15000 = 0$

or, $x^2 - 125x + 120x - 15000 = 0$

or, $x(x-125) + 120(x-125) = 0$

or, $(x-125)(x+120) = 0$

$\therefore\ x - 125 = 0$ or, $x + 120 = 0$

If $x - 125 = 0$, $x = 125$

Again, if $(x+120) = 0$, $x = -120$

Since the number of members i.e,, x cannot be negative, $x \neq -120$.

$\therefore\ x = 125$

Hence, number of members of the association is 125.

Example 2. Rafiq can do a work in 10 days and shafiq can do that work in 15 days. In how many days do they together finish the work ?

Solution : Let, Rafiq and Shafiq together can finish the wok in d days.

Let us make the following table :

Name	Number of days for doing the work	Part of the work done in 1 day	Work done in d days
Rafiq	10	$\frac{1}{10}$	$\frac{d}{10}$
Shafiq	15	$\frac{1}{15}$	$\frac{d}{15}$

By the questions, $\frac{d}{10}+\frac{d}{15}=1$

or, $d\left(\frac{1}{10}+\frac{1}{15}\right)=1$

or, $d\left(\frac{3+2}{30}\right)=1$

or, $\frac{5d}{30}=1$

or, $d=\frac{30}{5}=6$

$\therefore$ They together can finish the work in 6 days.

Example 3. A boatman can go x km in time t_1 hour against the current. To cover that distance along the current he takes t_2 hour. How much is the speed of the boat and the current.

Solution : Let the speed of the boat in still water be u km per hour and that of the current be v km per hour.

Then, along the current, the effective speed of boat is $(u+v)$ km per hour and against the current, the effective speed of boat is $(u-v)$ km per hour.

According to the question,

$$u+v=\frac{x}{t_2}\,.......(i)\quad [\because \text{speed}=\frac{\text{distance traversed}}{\text{time}}]$$

and $u-v=\frac{x}{t_1}\,.......(ii)$

Adding equations (i) and (ii) we get,

$$2u=\frac{x}{t_1}+\frac{x}{t_2}=x\left(\frac{1}{t_1}+\frac{1}{t_2}\right)$$

or, $u = \frac{x}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$

Subtracting equation (ii) from equation (i) we get,

$$2v = x\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

or, $v = \frac{x}{2}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$

Hence, speed of current is $\frac{x}{2}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$ km per hour

and speed of boat is $\frac{x}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$ km per hour.

Example 4. A pipe can fill up an empty tank in 12 minutes. Another pipe flows out 14 litre of water per minute. If the two pipes are opened together and the empty tank is filled up in 96 minutes, how much water does the tank contain ?

Solution : Let x litre of water flows in per minute by the first pipe and the tank can contain y litre of water.

According to the question, the tank is filled up by first pipe in 12 minutes,

$$\therefore \quad y = 12x \,.......(i)$$

Again, the empty tank is filled up by the two pipes together in 96 minutes.

$$\therefore \quad y = 96x - 96 \times 14 \,.........(ii)$$

From equation (i), we get, $x = \frac{y}{12}$

putting the value of x in equation (ii), we get,

$$y = 96 \times \frac{y}{12} - 96 \times 14$$

or, $y = 8y - 96 \times 14$

or, $7y = 96 \times 14$

or, $y = \frac{96 \times 14}{7} = 192$

Hence, total 192 litre of water is contained in the tank.

Activity :

1. For a picnic, a bus was hired at Tk. 2400 and it was decided that every passenger would have to give equal fare. But due to the absence of 10 passengers, fare per head was increased by Tk. 8. How many passengers did go by the bus and how much money did each of the passengers give as fare?
2. A and B together can do a work in p days. A alone can do that work in q days. In how many days can B alone do the work ?
3. A person rowing against the current can go 2 km per hour. If the speed of the current is 3 km per hour, how much time will he take to cover 32 km, rowing along the current ?

Example 5. Price of a book is Tk. 24.00. This price is 80% of the actual price. The Government subsidize the due price. How much money does the Govt. subsidize for each book ?

Solution : Market price = 80% of actual price

We know, $p = br$

Here, $p = \text{Tk. } 24$ and $r = 80\% = \frac{80}{100}$

$$\therefore\ 24 = b \times \frac{80}{100}$$

or, $b = \dfrac{\overset{6}{\cancel{24}} \times \overset{5}{\cancel{100}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{80}}}$ $\quad \therefore\ b = 30$

Hence, the actual price of the book is Tk. 30.

$\therefore$ amount of subsidized money = Tk. $(30 - 24)$

= Tk. 6.

$\therefore$ subsidized money for each book is Tk. 6.

Example 6. The loss is $r\%$ when n oranges are sold per taka. How many oranges are to be sold per taka to make a profit of $s\%$?

Solution : If the cost price is Tk. 100, the selling price at the loss of $r\%$ is Tk. $(100 - r)$.

If selling price is Tk. $(100 - r)$, cost price is Tk. 100

$\therefore$,, ,, ,, ,, Tk. 1 ,, ,, ,, Tk. $\frac{100}{100 - r}$

Again, if cost price is Tk. 100, selling price at the profit of $s\%$ is Tk. $(100+s)$

$\therefore$,, ,, ,, ,, Tk. 1 ,, ,, ,, ,, ,, ,, ,, ,, Tk. $\frac{100+s}{100}$

$\therefore$,, ,, ,, ,, Tk. $\frac{100}{100-r}$,, ,, ,, ,, ,, ,, ,, Tk. $\left(\frac{100+s}{100}\times\frac{100}{100-r}\right)$

$= \text{Tk. } \frac{100+s}{100-r}$

$\therefore$ in Tk. $\frac{100+s}{100-r}$, number of oranges is to be sold = n

$\therefore$ in Tk. 1, ,, ,, ,, ,, ,, ,, ,, $n\times\left(\frac{100-r}{100+s}\right)$

Hence, $\frac{n(100-r)}{100+s}$ oranges are to be sold per taka.

Example 7. What is the profit of Tk. 650 in 6 years at the rate of profit Tk. 7 percent per annum ?

Solution : We know , $I = Pnr$.

Here, $P =$ Tk. 650, $n = 6$, $s = 7$

$\therefore \quad r = \frac{s}{100} = \frac{7}{100}$

$\therefore \quad I = 650\times 6\times\frac{7}{100} = 273$

Hence, profit is Tk. 273.

Example 8. Find the compound principal and compound profit of Tk. 15000 in 3 years at the profit of 6 percent per annum.

Solution : We know, $C = P(1+r)^n$, where C is the profit principal in the case of compound profit.

Given, $P = \text{Tk.}15000$, $r = 6\% = \frac{6}{100}$, $n = 3$ years

$\therefore C = 15000\left(1+\frac{6}{100}\right)^3 = 15000\left(1+\frac{3}{50}\right)^3$

$= 15000\left(\frac{53}{50}\right)^3$

$$= \frac{\overset{3}{\cancel{15}} \times 53 \times 53 \times 53}{\cancel{125}_{25}} = \frac{3 \times 148877}{25}$$

$$= \frac{446631}{25} = 17865 \cdot 24$$

$\therefore$ compound principal is Tk. $17865 \cdot 24$

$\therefore$ compound profit = Tk. $(17865 \cdot 24 - 15000)$

= Tk. $2865 \cdot 24$

Activity :

1. The loss is $n\%$ when 10 lemons are sold per taka. How many lemons are to be sold per taka to make the profit of $z\%$?
2. What will be the profit principal of Tk. 750 in 4 years at the rate of simple profit $6\frac{1}{2}$ percent per annum ?
3. Find the compound principal of Tk. 2000 in 3 years at the rate of compound profit of Tk. 4 percent per annum.

Exercise 3·5

1. Which one of the following is the factorized form of $x^2 - 7x + 6$?

   (a) $(x-2)(x-3)$ (b) $(x-1)(x+8)$

   (c) $(x-1)(x-6)$ (d) $(x+1)(x+6)$

2. If $f(x) = x^2 - 4x + 4$, which one of the following is the value of $f(2)$?

   (a) 4 (b) 2

   (c) 1 (d) 0

3. If $x + y = x - y$, which one of the following is the value of y ?

   (a) -1 (b) 0

   (c) 1 (d) 2

4. Which one of the following is the lowest form of $\frac{x^2 + 3x^3}{x + 3x^2}$?

   (a) x^2 (b) x

   (c) 1 (d) 0

5. Which one of the following is the lowest form of $\frac{1 - x^2}{1 - x}$?

   (a) 1 (b) x

6. Which one of the following is the value of $\frac{1}{2}\{(a+b)^2-(a-b)^2\}$?

(a) $2(a^2+b^2)$ (b) a^2+b^2

(c) $2ab$ (d) $4ab$

7. If $x+\frac{2}{x}=3$, what is the value of $x^3+\frac{8}{x^3}$?

(a) 1 (b) 8 (c) 9 (d) 16

8. Which one of the following is the factorized form of p^4+p^2+1 ?

(a) $(p^2-p+1)(p^2+p-1)$ (b) $(p^2-p-1)(p^2+p+1)$

(c) $(p^2+p+1)(p^2+p+1)$ (d) $(p^2+p+1)(p^2-p+1)$

9. What are the factors of x^2-5x+4?

(a) $(x-1),(x-4)$ (b) $(x+1),(x-4)$

(c) $(x+2),(x-2)$ (d) $(x-5),(x-1)$

10. What is the value of $(x-7)(x-5)$?

(a) $x^2+12x+35$ (b) $x^2+12x-35$

(c) $x^2-12x+35$ (d) $x^2-12x-35$

11. What is the value of $\frac{2\cdot9\times2\cdot9-1\cdot1\times1\cdot1}{2\cdot9-1\cdot1}$?

(a) $1\cdot8$ (b) $1\cdot9$

(c) 2 (d) 4

12. If $x=2-\sqrt{3}$, what is the value of x^2 ?

(a) 1 (b) $7-4\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $\frac{1}{2-\sqrt{3}}$

13. If $f(x)=x^2-5x+6$ and $f(x)=0$, x = what ?

(a) 2, 3 (b) $-5, 1$

(c) $-2, 3$ (d) 1, 5

14.

	x	$+6$
x	x^2	$+6x$
-5	$-5x$	-30

Which one of the following is the total area of the table above ?

(a) $x^2 - 5x + 30$ (b) $x^2 + x - 30$

(c) $x^2 + 6x - 30$ (d) $x^2 - x + 30$

15. A can do a work in x days, B can do that work in $3x$ days. In the same time, how many times does A of B work ?

(a) 2 times (b) $2\frac{1}{2}$ times

(c) 3 times (d) 4 times

16. If $a + b = -c$, $a^2 + 2ab + b^2$ is expressed in terms of c, which one of the following will be ?

(a) $-c^2$ (b) c^2 (c) bc (d) ca

17. If $x + y = 3, xy = 2$, what is the value of $x^3 + y^3$?

(a) 9 (b) 18

(c) 19 (d) 27

18. Which one is the factorized form of $8x^3 + 27y^3$?

(a) $(2x - 3y)(4x^2 + 6xy + 9y^2)$ (b) $(2x + 3y)(4x^2 - 6xy + 9y^2)$

(c) $(2x - 3y)(4x^2 - 9y^2)$ (d) $(2x + 3y)(4x^2 + 9y^2)$

19. What is to be added to $9x^2 + 16y^2$, so that their sum will be a perfect square ?

(a) $6xy$ (b) $12xy$

(c) $24xy$ (d) $144xy$

20. If $x - y = 4$, which one of the following statements is correct ?

(a) $x^3 - y^3 - 4xy = 64$ (b) $x^3 - y^3 - 12xy = 12$

(c) $x^3 - y^3 - 3xy = 64$ (d) $x^3 - y^3 - 12xy = 64$

21. If $x^4 - x^2 + 1 = 0$,

(1) $x^2 + \frac{1}{x^2}$ = what ?

(a) 4 (b) 2 (c) 1 (d) 0

(2) What is the value of $\left(x + \frac{1}{x}\right)^2$?

(a) 4 (b) 3

(c) 2 (d) 1

(3) $x^3 + \frac{1}{x^3} =$ what ?

(a) 3 (b) 2

(c) 1 (d) 0

22. A can do a work in p days and B can do it in $2p$ days. They started to do the work together and after some days A left the work unfinished. B completed the rest of the work in r days. In how many days was the work finished ?
23. 50 persons can do a work in 12 days by working 8 hours a day. Working how many hours per day can 60 persons finish the work in 16 days. ?
24. Mita can do a work in x days and Rita can do that work in y days. In how many days will they together complete the work ?
25. A bus was hired at Tk. 57000 to go for a picnic under the condition that every passenger would bare equal fare. But due to the absence of 5 passengers, the fare was increased by Tk. 3 per head. How many passengers availed the bus ?
26. A boatman can go d km in p hours against the current. He takes q hours to cover that distance along the current. What is the speed of the current and the boat ?
27. A boatman plying by oar goes 15 km and returns from there in 4 hours. He goes 5 km at a period of time along the current and goes 3 km at the same period of time against the current. Find the speed of the oar and current.
28. Two pipes are connected with a tank. The empty tank is filled up in t_1 minutes by the first pipe and it becomes empty in t_2 minutes by the second pipe. If the two pipes are opened together, in how much time will the tank be filled up ? (here $t_1 > t_2$)
29. A tank is filled up in 12 minutes by a pipe. Another pipe flows out 15 litre of water in 1 minute. When the tank remains empty, the two pipes are opened togethr and the tank is filled up in 48 minutes. How much water does the tank contain ?
30. If a pen is sold at Tk. 11, there is a profit of 10%. What was the cost price of the pen ?
31. Due to the sale of a notebook at Tk. 36, there was a loss. If the notebook would be sold at Tk. 72, there would be profit amounting twice the loss. What was the cost price of the notebook ?
32. Divide Tk. 260 among *A, B* and *C* in such a way that 2 times the share of *A*, 3 times the share of *B* and 4 times the share of *C* are equal to one another.
33. Due to the selling of a commodity at the loss of $x\%$ such price is obtained that due to the selling at the profit of $3x\%$ Tk.$18x$ more is obtained. What was the cost price of the commodity ?

34. If the simple profit of Tk. 300 in 4 years and the simple profit of Tk. 400 in 5 years together are Tk. 148, what is the percentage of profit ?
35. If the difference of simple profit and compound profit of some principal in 2 years is Tk. 1 at the rate of profit 4%, what is the principal ?
36. Some principal beomes Tk. 460 with simple profit in 3 years and Tk. 600 with simple profit in 5 years. What is the rate of profit ?
37. How much money will become Tk. 985 as profit principal in 13 years at the rate of simple profit 5% per annum ?
38. How much money will become Tk. 1248 as profit principal in 12 years at the rate of profit 5% per annum ?
39. Find the difference of simple profit and compound profit of Tk. 8000 in 3 years at the rate of profit 5%.
40. The Value Added Tax (VAT) of sweets is $x\%$. If a trader sells sweets at Tk. P including VAT, how much VAT is he to pay ? If $x=15$, $P=2300$, what is the amount of VAT ?
41. Sum of a number and its multiplicative inverse is 3.

    (a) Taking the number as the variable x, express the above information by an equation.

    (b) Find the value of $x^3 - \frac{1}{x^3}$.

    (c) Prove that, $x^5 - \frac{1}{x^5} = 123$.
42. Each of the members of an association decided to subscribe 100 times the number of members. But 7 members did not subscribe. As a result, amount of subscription for each member was increased by Tk. 500 than the previous.

    (a) If the number of members is x and total amount of subscription is Tk. A, find the relation between them.

    (b) Find the number of members of the association and total amount of subscription.

    (c) $\frac{1}{4}$ of total amount of subscription at the rate of simple profit 5% and rest of the money at the rate of simple profit 4% were invested for 2 years. Find the total profit.

Chapter Four

Exponents and Logarithms

Very large or very small numbers or expressions can easily be expressed in writing them by exponents. As a result, calculations and solution of mathematical problems become easier. Scientific or standard form of a number is expressed by exponents. Therefore, every student should have the knowledge about the idea of exponents and its applications.

Exponents beget logarithms. Multiplication and division of numbers or expressions and exponent related calculations have become easier with the help of logarithms. Use of logarithm in scientific calculations was the only way before the practice of using the calculator and computer at present. Still the use of logarithm is important as the alternative of calculator and computer. In this chapter, exponents and logarithms have been discussed in detail.

At the end of the chapter, the students will be able to –

- Explain the rational exponent
- Explain and apply the positive integral exponents, zero and negative integral exponents
- Solve the problems by describing and applying the rules of exponents
- Explain the n th root and rational fractional exponents and express the n th root in terms of exponents
- Explain the logarithms
- Prove and apply the formulae of logarithms
- Explain the natural logarithm and common logarithm
- Explain the scientific form of numbers
- Explain the characteristic and mantissa of common logarithm and
- Find common and natural logarithm by calculator.

4.1 Exponents or Indices

In class VI, we have got the idea of exponents and in class VII, we have known the exponential rules for multiplication and division.

Expression associated with exponent and base is called exponential expression.

Activity : Fill in the blanks

Successive multiplication of the same number or expression	**Exponential expression**	**Base**	**Power or exponent**
$2\times2\times2$	2^3	2	3
$3\times3\times3\times3$		3	
$a\times a\times a$	a^3		
$b\times b\times b\times b\times b$			5

If a is any real number, successive multiplication of n times a; that is, $a \times a \times a \times ..... \times a$ is written in the form a^n, where n is a positive integer.
$a \times a \times a \times ..... \times a$ (n times a) = a^n.

Here $n \rightarrow$ index or power
$a \rightarrow$ base

Again, conversely, a^n = $a \times a \times a \times ..... \times a$ (n times a) . Exponents may not only be positive integer, it may also be negative integer or positive fraction or negative fraction. That is, for $a \in R$ (set of real numbers) and $n \in Q$ (set of rational numbers), a^n is defined. Besides, it may also be irrational exponent. But as it is out of curriculum, it has not been discussed in this chapter.

4.2 Formulae for exponents

Let, $a \in R; m, n \in N$.

Formula 1. $a^m \times a^n = a^{m+n}$

Formula 2. $\frac{a^m}{a^n} = \begin{cases} a^{m-n}, & \text{when } m>n \\ \frac{1}{a^{n-m}}, & \text{when } n>m, a\neq 0 \end{cases}$

Fill in the blanks of the following table :

a^m, a^n $a \neq 0$	$m > n$	$n > m$
	$m = 5,\ n = 3$	$m = 3,\ n = 5$
$a^m \times a^n$	$a^5 \times a^3 = (a \times a \times a \times a \times a) \times (a \times a \times a)$ $= a \times a \times a \times a \times a \ \times a \times a \times a$ $= a^8 = a^{5+3}$	$a^3 \times a^5 =$
$\frac{a^m}{a^n}$	$\frac{a^5}{a^3} =$	$\frac{a^3}{a^5} = \frac{a \times a \times a}{a \times a \times a \times a \times a}$ $= a^{\frac{1}{2}} = a^{\frac{1}{5-3}}$

$\therefore\ a^m \times a^n = a^{m+n}$

and $\frac{a^m}{a^n} = \begin{cases} a^{m-n}, & \text{when } m>n \\ \frac{1}{a^{n-m}}, & \text{when } n>m \end{cases}$

Formula 3. $(ab)^n = a^n \times b^n$

We observe, $(5 \times 2)^3 = (5 \times 2) \times (5 \times 2) \times (5 \times 2)$ $[\because a^3 = a \times a \times a; a = 5 \times 2]$

$= 5 \times 2 \times 5 \times 2 \times 5 \times 2$

$= (5 \times 5 \times 5) \times (2 \times 2 \times 2)$

$= 5^3 \times 2^3$

In general, $(ab)^n = ab \times ab \times ab \times .... \times ab$ [Successive multiplication of n times ab]

$= (a \times a \times a \times .... \times a) \times (b \times b \times b \times .... \times b)$

$= a^n b^n$

Formula 4. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, (b \neq 0)$

We observe, $\left(\frac{5}{2}\right)^3 = \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} = \frac{5 \times 5 \times 5}{2 \times 2 \times 2} = \frac{5^3}{2^3}$

In general, $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times .... \times \frac{a}{b}$ [Successive multiplication of n times $\frac{a}{b}$]

$= \frac{a \times a \times a \times ...... \times a}{b \times b \times b \times ...... \times b} = \frac{a^n}{b^n}$

Formula 5. $a^0 = 1, (a \neq 0)$

We have, $\frac{a^n}{a^n} = a^{n-n} = a^0$

Again, $\frac{a^n}{a^n} = \frac{a \times a \times a \times ..... \times a}{a \times a \times a \times ..... \times a}$ [in both the cases of num. and den multiplication of n times a]

$= 1$

$\therefore\ a^0 = 1.$

Formula 6. $a^{-n} = \frac{1}{a^n}, (a \neq 0)$

$a^{-n} = \frac{a^{-n} \times a^n}{1 \times a^n}$ [multiplying both num. and denom. by a^n]

$= \frac{a^{-n+n}}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}$

$\therefore\ a^{-n} = \frac{1}{a^n}$

Remark : $\frac{1}{a^n} = \frac{a^o}{a^n} = a^{o-n} = a^{-n}$

Formula 7. $(a^m)^n = a^{mn}$

$(a^m)^n = a^m \times a^m \times a^m \times .... \times a^m$ [successive multiplication of n times a^m]

$= a^{m+m+m+.........+m}$ [in the power, sum of n times of exponent m]

$= a^{n\times m} = a^{mn}$

$\therefore\ (a^m)^n = a^{mn}$

Example 1. Find the values (a) $\frac{5^2}{5^3}$ (b) $\left(\frac{2}{3}\right)^5 \times \left(\frac{2}{3}\right)^{-5}$

Solution : (a) $\frac{5^2}{5^3} = 5^{2-3} = 5^{-1} = \frac{1}{5^1} = \frac{1}{5}$

(b) $\left(\frac{2}{3}\right)^5 \times \left(\frac{2}{3}\right)^{-5} = \left(\frac{2}{3}\right)^{5-5} = \left(\frac{2}{3}\right)^0 = 1$

Example 2. Simplify :(a) $\frac{5^4 \times 8 \times 16}{2^5 \times 125}$ (b) $\frac{3\cdot 2^n - 4\cdot 2^{n-2}}{2^n - 2^{n-1}}$

Solution :(a) $\frac{5^4 \times 8 \times 16}{2^5 \times 125} = \frac{5^4 \times 2^3 \times 2^4}{2^5 \times 5^3} = \frac{5^4 \times 2^{3+4}}{5^3 \times 2^5} = \frac{5^4}{5^3} \times \frac{2^7}{2^5} = 5^{4-3} \times 2^{7-5}$

$= 5^1 \times 2^2 = 5 \times 4 = 20$

(b) $\frac{3\cdot 2^n - 4\cdot 2^{n-2}}{2^n - 2^{n-1}} = \frac{3\cdot 2^n - 2^2\cdot 2^{n-2}}{2^n - 2^n\cdot 2^{-1}} = \frac{3\cdot 2^n - 2^{2+n-2}}{2^n - 2^n\cdot\frac{1}{2}}$

$= \frac{3\cdot 2^n - 2^n}{\left(1-\frac{1}{2}\right)\cdot 2^n} = \frac{(3-1)\cdot 2^n}{\frac{1}{2}\cdot 2^n} = \frac{2\cdot 2^n}{\frac{1}{2}\cdot 2^n} = 2\cdot 2 = 4.$

Example 3. Show that $(a^p)^{q-r}\cdot(a^q)^{r-p}\cdot(a^r)^{p-q} = 1$

Solution : $(a^p)^{q-r}\cdot(a^q)^{r-p}\cdot(a^r)^{p-q}$

$= a^{p(q-r)}\cdot a^{q(r-p)}\cdot a^{r(p-q)}$ $[\because (a^m)^n = a^{mn}]$

$= a^{pq-pr}\cdot a^{qr-pq}\cdot a^{pr-qr}$

$= a^{pq-pr+qr-pq+pr-qr}$

$= a^0 = 1.$

Activity : Fill in the blank boxes :

(*i*) $3\times3\times3\times3 = 3^{\square}$ (*ii*) $5^{\square}\times 5^3 = 5^5$ (*iii*) $a^2 \times a^{\square} = a^{-3}$ (*iv*) $\frac{4}{4^{\square}} = 1^{\square}$

(*v*) $(-5)^0 = \square$

4.3 n th root

We notice, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = \left(5^{\frac{1}{2}}\right)^2$

Again, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

$\therefore \left(5^{\frac{1}{2}}\right)^2 = 5.$

Square (power 2) of $5^{\frac{1}{2}} = 5$ and square root (second root) of 5 = $5^{\frac{1}{2}}$

$5^{\frac{1}{2}}$ is written as $\sqrt{5}$ in terms of the sign $\sqrt{\ }$ of square root.

Again, we notice $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = \left(5^{\frac{1}{3}}\right)^3$

Again, $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5$

$\therefore \left(5^{\frac{1}{3}}\right)^3 = 5.$

Cube (power 3) of $5^{\frac{1}{3}} = 5$ and cube root (third root) of 5 = $5^{\frac{1}{3}}$.

$5^{\frac{1}{3}}$ is written as $\sqrt[3]{5}$ in terms of the sign $\sqrt[3]{\ }$ of cube root.

In the case of n th root,

$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots. \times a^{\frac{1}{n}}$ [successive multiplication of n times $a^{\frac{1}{n}}$]

$= \left(a^{\frac{1}{n}}\right)^n.$

Again, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots. \times a^{\frac{1}{n}}$

$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\ldots\ldots+\frac{1}{n}}$ [in the exponent, sum of n times $\frac{1}{n}$]

$= a^{n\times\frac{1}{n}} = a$

$\therefore \left(a^{\frac{1}{n}}\right)^n = a.$

n th power of $a^{\frac{1}{n}} = a$ and n th root of $a = a^{\frac{1}{n}}$

i.e. n th power of $a^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^n = a$ and n th root of $a = (a)^{\frac{1}{n}} = a^{\frac{1}{n}} = \sqrt[n]{a}$. n th root of a is written as $\sqrt[n]{a}$.

Example 4. Simplify :(a) $7^{\frac{3}{4}} \cdot 7^{\frac{1}{2}}$ (b) $(16)^{\frac{3}{4}} \div (16)^{\frac{1}{2}}$ (c) $\left(10^{\frac{2}{3}}\right)^{\frac{3}{4}}$

Solution : (a) $7^{\frac{3}{4}} \cdot 7^{\frac{1}{2}} = 7^{\frac{3}{4}+\frac{1}{2}} = 7^{\frac{5}{4}}$

(b) $(16)^{\frac{3}{4}} \div (16)^{\frac{1}{2}} = \dfrac{(16)^{\frac{3}{4}}}{(16)^{\frac{1}{2}}} = (16)^{\frac{3}{4}-\frac{1}{2}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = (2)^{4\times\frac{1}{4}} = 2.$

(c) $\left(10^{\frac{2}{3}}\right)^{\frac{3}{4}} = 10^{\frac{2}{3}\times\frac{3}{4}} = 10^{\frac{1}{2}} = \sqrt{10}.$

Example 5. Simplify :(a) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54}$ (b) $(-3)^3 \times \left(-\frac{1}{2}\right)^2$

Solution : (a) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54} = \dfrac{1}{(12)^{\frac{1}{2}}} \times (54)^{\frac{1}{3}}$

$= \dfrac{1}{(2^2 \times 3)^{\frac{1}{2}}} \times (3^3 \times 2)^{\frac{1}{3}} = \dfrac{1}{(2^2)^{\frac{1}{2}} \cdot \times 3^{\frac{1}{2}}} \times (3^3)^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$

$= \dfrac{1}{(2 \cdot 3)^{\frac{1}{2}}} \times (3 \cdot 2)^{\frac{1}{3}} = \dfrac{2^{\frac{1}{3}}}{2^1} \times \dfrac{3^1}{3^{\frac{1}{2}}} = \dfrac{3^{1-\frac{1}{2}}}{2^{1-\frac{1}{3}}} = \dfrac{3^{\frac{1}{2}}}{2^{\frac{2}{3}}} = \dfrac{3^{\frac{1}{2}}}{4^{\frac{1}{3}}} = \dfrac{\sqrt{3}}{\sqrt[3]{4}}.$

(b) $(-3)^3 \times \left(-\frac{1}{2}\right)^2 = (-3)(-3)(-3) \times \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$= -27 \times \frac{1}{4} = -\frac{27}{4}$

Activity : Simplify : (i) $\dfrac{2^4 \cdot 2^2}{32}$ (ii) $\left(\frac{2}{3}\right)^{\frac{5}{2}} \times \left(\frac{2}{3}\right)^{\frac{-5}{2}}$ (iii) $8^{\frac{3}{4}} \div 8^{\frac{1}{2}}$

To be noticed :

1. Under the condition $a > 0, a \neq 1$, if $a^x = a^y$, $x = y$
2. Under the condition $a > 0, b > 0, x \neq 0$, if $a^x = b^x$, $a = b$

Example 6. Solve : $4^{x+1} = 32$.

Solution : $4^{x+1} = 32$

or $(2^2)^{x+1} = 32$, or, $2^{2x+2} = 2^5$ [if $a^x = a^y$, $x = y$]

$\therefore 2x + 2 = 5,$

or, $2x = 5 - 2$, or, $2x = 3$

$\therefore x = \frac{3}{2}$

$\therefore$ Solution is $x = \frac{3}{2}$

Exercise 4.1

Simplify (1 – 10) :

1. $\frac{3^3 \cdot 3^5}{3^6}$ 2. $\frac{5^3 \cdot 8}{2^4 \cdot 125}$ 3. $\frac{7^3 \times 7^{-3}}{3 \times 3^{-4}}$ 4. $\frac{\sqrt[3]{7^2} \cdot \sqrt[3]{7}}{\sqrt{7}}$

5. $(2^{-1} + 5^{-1})^{-1}$ 6. $(2a^{-1} + 3b^{-1})^{-1}$ 7. $\left(\frac{a^2 b^{-1}}{a^{-2} 6}\right)^2$

8. $\sqrt{x^{-1}y} \cdot \sqrt{y^{-1}z} \cdot \sqrt{z^{-1}x}, (x > 0, y > 0, z > 0)$ 9. $\frac{2^{n+4} - 4 \cdot 2^{n+1}}{2^{n+2} \div 2}$ 10. $\frac{3^{m+1}}{(2^m)^{m-1}} \div \frac{3^{m+1}}{(3^{m-1})^{m+1}}$

Prove (11 – 18) :

11. $\frac{4^n - 1}{2^n - 1} = 2^n + 1$ 12. $\frac{2^{p+1} \cdot 3^{2p-q} \cdot 5^{p+q} \cdot 6^p}{6^q \cdot 10^{p+2} \cdot 15^q} = \frac{1}{50}$

13. $\left(\frac{a^\ell}{a^m}\right)^n \cdot \left(\frac{a^m}{a^n}\right)^\ell \cdot \left(\frac{a^n}{a^\ell}\right)^m = 1$ 14. $\frac{a^{p+q}}{a^{2r}} \times \frac{a^{q+r}}{a^{2p}} \times \frac{a^{r+p}}{a^{2q}} = 1$

15. $\left(\frac{x^a}{x^b}\right)^{\frac{1}{ab}} \cdot \left(\frac{x^b}{x^c}\right)^{\frac{1}{bc}} \cdot \left(\frac{x^c}{x^a}\right)^{\frac{1}{ca}} = 1$ 16. $\left(\frac{x^a}{x^b}\right)^{a+b} \cdot \left(\frac{x^b}{x^c}\right)^{b+c} \cdot \left(\frac{x^c}{x^a}\right)^{c+a} = 1$

17. $\left(\frac{x^p}{x^q}\right)^{p+q-r} \times \left(\frac{x^q}{x^r}\right)^{q+r-p} \times \left(\frac{x^r}{x^p}\right)^{r+p-q} = 1$

18. If $a^x = b$, $b^y = c$ and $c^z = a$, show that $xyz = 1$

Solve (19 – 22) :

19. $4^x = 8$ 20. $2^{2x+1} = 128$ 21. $\left(\sqrt{3}\right)^{x+1} = \left(\sqrt[3]{3}\right)^{2x-1}$ 22. $2^x + 2^{1-x} = 3$

4.4 Logarithm

Logarithm is used to find the values of exponential expressions. Logarithm is written in brief as '*Log*'. Product, quotient, etc. of large numbers or quantities can easily be determined by the help of log.

We know, $2^3 = 8$; this mathematical statement is written in terms of log as $\log_2 8 = 3$. Again, conversely, if $\log_2 8 = 3$, it can be written in terms of exponents as $2^3 = 8$. That is, if $2^3 = 8$, then $\log_2 8 = 3$ and conversely, if $\log_2 8 = 3$, then $2^3 = 8$. Similarly, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ can be written in terms of log as $\log_2 \frac{1}{8} = -3$.

If $a^x = N$, ($a > 0, a \neq 1$), $x = \log_a N$ is defined as a based $\log N$.

To be noticed : Whatever may be the values of x, positive or negative, a^x is always positive. So, only the log of positive numbers has values which are real ; log of zero or negative numbers have no real value.

Activity-1 : Express in terms of log :	**Activity-2 :** Fill in the blanks :	
(i) $10^2 = 100$	in terms of exponent	in terms of log
(ii) $3^{-2} == \frac{1}{9}$	$10^0 = 1$	$\log_{10} 1 = 0$
(iii) $2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$	$e^0 = ...$	$\log_e 1 = ...$
(iv) $\sqrt{2^4}$	$a^0 = 1$	...... = ...
	$10^1 = 10$	$\log_{10} 10 = 1$
	$e^1 = ...$	.... = ...
	.... = ...	$\log_a a = 1$

Formulae of Logarithms :

Let, $a > 0, a \neq 1; b > 0, b \neq 1$ and $M > 0, N > 0$.

Formula 1. (a) $\log_a 1 = 0, (a > 0, a \neq 1)$

(b) $\log_a a = 1, (a > 0, a \neq 1)$

Proof : (a) We know from the formula of exponents, $a^0 = 1$

$\therefore$ from the definition of log, we get, $\log_a 1 = 0$ (proved)

(b) We know, from the formula of exponents, $a^1 = a$

$\therefore$ from the definition of log, we get, $\log_a a = 1$ (proved).

Formula 2. $\log_a (MN) = \log_a M + \log_a N$

Proof : Let, $\log_a M = x, \log_a N = y$;

$\therefore\ M = a^x, N = a^y$

Now, $MN = a^x \cdot a^y = a^{x+y}$

$\therefore \log_a(MN) = x + y$, or $\log_a(MN) = \log_a M + \log_a N$ [putting the values of x, y]

$\therefore \log_a(MN) = \log_a M + \log_a N$. (proved)

Note 1. $\log_a(MNP.....) = \log_a M + \log_a N + \log_a P + ...$

Note 2. $\log_a(M \pm N) \neq \log_a M \pm \log_a N$

Formula 3. $\log_a \frac{M}{N} = \log_a M - \log_a N$

Proof : Let $\log_a M = x, \log_a N = y$;

$\therefore M = a^x, N = a^y$

Now, $\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$

$\therefore \log_a\left(\frac{M}{N}\right) = x - y$

$\therefore \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ (proved).

Formula 4. $\log_a M^r = r \log_a M$.

Proof : :Let $\log_a M = x$; $\therefore M = a^x$

$\therefore (M)^r = (a^x)^r$; or $M^r = a^{rx}$

$\therefore \log_a M^r = rx$; or $\log_a M^r = r \log_a M$

$\therefore \log_a M^r = r \log_a M$. (proved).

N.B. : $(\log_a M)^r \neq r \log_a M$

Formula 5. $\log_a$ M = $\log_b$ M × $\log_a$ b, (change of base)

Proof : Let, $\log_a M = x, \log_b M = y$

$\therefore a^x = M, b^y = M$

$\therefore a^x = b^y$, or $(a^x)^{\frac{1}{y}} = (b^y)^{\frac{1}{y}}$

or $b = a^{\frac{x}{y}}$

$\therefore \frac{x}{y} = \log_a b$, or $x = y \log_a b$

or , $x = y \log_a b$, or $\log_a M = \log_b M \times \log_a b$ (proved).

Corollary : $\log_a b = \frac{1}{\log_b a}$, or, $\log_b a = \frac{1}{\log_a b}$

Proof : We know, $\log_a M = \log_b M \times \log_a b$ [formula 5]

Putting $M = a$ we get,

$\log_a a = \log_b a \times \log_a b$

or $1 = \log_b a \times \log_b b$;

$\therefore \log_b a = \dfrac{1}{\log_a b}$

or $\log_a b = \dfrac{1}{\log_b a}$ (proved).

Example 7. Find the value : (a) $\log_{10} 100$ (b) $\log_3\left(\dfrac{1}{9}\right)$ (c) $\log_{\sqrt{3}} 81$

Solution:

(a) $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10$ $[\because \log_{10} M^r = r\log_{10} M]$

$= 2 \times 1$ $[\because \log_a a = 1]$ $= 2$

(b) $\log_3\left(\dfrac{1}{9}\right) = \log_3\left(\dfrac{1}{3^2}\right) = \log_3 3^{-2} = -2\log_3 3$ $[\because \log_a M^r = r\log_a M]$

$= -2 \times 1 [\because \log_a a = 1] = -2$

(c) $\log_{\sqrt{3}} 81 = \log_{\sqrt{3}} 3^4 = \log_{\sqrt{3}}\{(\sqrt{3})^2\}^4 = \log_{\sqrt{3}} (\sqrt{3})^8$

$= 8\log_{\sqrt{3}} \sqrt{3} [\because \log_a M^r = r\log_a M]$

$= 8 \times 1, [\because \log_a a = 1]$

$= 8$

Example 8. (a) What is the log of $5\sqrt{5}$ to the base 5 ?

(b) $\log 400 = 4$; what is the base ?

Solution : (a) $5\sqrt{5}$ to the base 5

$= \log_5 5\sqrt{5} = \log_5 (5 \times 5^{\frac{1}{2}}) = \log_5 5^{\frac{3}{2}}$

$= \dfrac{3}{2}\log_5 5, [\because \log_a M^r = r\log_a M]$

$= \dfrac{3}{2} \times 1, [\because \log_a a = 1]$

$= \dfrac{3}{2}$

(b) Let the base be a.

$\therefore$ by the question, $\log_a 400 = 4$

$\therefore$ $a^4 = 400$

or $a^4 = (20)^2 = \{(2\sqrt{5})^2\}^2 = (2\sqrt{5})^4$

or $a^4 = (2\sqrt{5})^4$

$\therefore\ a = 2\sqrt{5}$ [$\because$ if $a^x = b^x$, $a = b$]

$\therefore$ the base is $2\sqrt{5}$

Example 9. Find the value of x:

(a) $\log_{10} x = -2$ (b) $\log_x 324 = 4$

Solution :

(a) $\log_{10} x = -2$

$\therefore x = 10^{-2}$

$\therefore\ x = \frac{1}{10^2} = \frac{1}{100} = 0\cdot 01$

$\therefore\ x = 0\cdot 01$

(b) $\log_x 324 = 4$

$\therefore x^4 = 324 = 3\times 3\times 3\times 3\times 2\times 2$

$= 3^4 \times 2^2 = 3^4 \times (\sqrt{2})^4$

or $x^4 = (3\sqrt{2})^4$

$\therefore x = 3\sqrt{2}$.

Example 10. Prove that, $3\log_{10} 2 + \log_{10} 5 = \log_{10} 40$

Solution :Left hand side = $3\log_{10} 2 + \log_{10} 5$

$= \log_{10} 2^3 + \log_{10} 5, [\because \log_a M^r = r\log_a M]$

$= \log_{10} 8 + \log_{10} 5$

$= \log_{10}(8\times 5), [\because \log_a(MN) = \log_a M + \log_a N]$

$= \log_{10} 40$ = Right hand side (proved).

Example 11. Simplify : $\frac{\log_{10}\sqrt{27} + \log_{10} 8 - \log_{10}\sqrt{1000}}{\log_{10} 1\cdot 2}$

Solution : $\frac{\log_{10}\sqrt{27} + \log_{10} 8 - \log_{10}\sqrt{1000}}{\log_{10} 1\cdot 2}$

$$= \frac{\log_{10}(3^3)^{\frac{1}{2}} + \log_{10} 2^3 - \log_{10}(10^3)^{\frac{1}{2}}}{\log_{10}\frac{12}{10}}$$

$$= \frac{\log_{10} 3^{\frac{3}{2}} + \log_{10} 2^3 - \log_{10} 10^{\frac{3}{2}}}{\log_{10} 12 - \log_{10} 10}$$

$$= \frac{\frac{3}{2}\log_{10} 3 + 3\log_{10} 2 - \frac{3}{2}\log_{10} 10}{\log_{10}(3\times 2^2) - \log_{10} 10}$$

$$= \frac{\frac{3}{2}(\log_{10} 3 + 2\log_{10} 2 - 1)}{(\log_{10} 3 + 2\log_{10} 2 - 1)} \qquad [\because \log_{10} 10 = 1]$$

$$= \frac{3}{2}.$$

Exercise 4.2

1. Find the value :(a) $\log_3 81$ (b) $\log_5 \sqrt[3]{5}$ (c) $\log_4 2$ (d) $\log_{2\sqrt{5}} 400$
   (e) $\log_5 \left(\sqrt[3]{5} \cdot \sqrt{5}\right)$
2. Find the value of x :(a) $\log_5 x = 3$ (b) $\log_x 25 = 2$ (c) $\log_x \frac{1}{16} = -2$
3. Show that,
   (a) $5\log_{10} 5 - \log_{10} 25 = \log_{10} 125$
   (b) $\log_{10} \frac{50}{147} = \log_{10} 2 + 2\log_{10} 5 - \log_{10} 3 - 2\log_{10} 7$
   (c) $3\log_{10} 2 + 2\log_{10} 3 + \log_{10} 5 = \log_{10} 360$
4. Simplify :
   (a) $7\log_{10} \frac{10}{9} - 2\log_{10} \frac{25}{24} + 3\log_{10} \frac{81}{80}$
   (b) $\log_7 \left(\sqrt[5]{7} \cdot \sqrt{7}\right) - \log_3 \sqrt[3]{3} + \log_4{}^2$
   (c) $\log_e \frac{a^3b^3}{c^3} + \log_e \frac{b^3c^3}{d^3} + \log_e \frac{c^3d^3}{a^3} - 3\log_e b^2c$

4.5 Scientific or Standard form of numbers

We can express very large numbers or very small numbers in easy and small form by exponents.

Such as, velocity of light $= 300000$ km/sec $= 300000000$ m/sec

$= 3 \times 100000000$ m/sec. $= 3 \times 10^8$ m/sec.

Again, radius of a hydrogen atom $= 0 \cdot 00000000037$ cm

$= \frac{37}{10000000000}$ cm $= 37 \times 10^{-10}$ cm

$= 3 \cdot 7 \times 10 \times 10^{-10}$ cm $= 3 \cdot 7 \times 10^{-9}$ cm

For convenience, very large number or very small number is expressed in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in Z$. The form $a \times 10^n$ of any number is called the scientific or standard form of the number.

Activity : Express the following numbers in scientific form :	
(a) 15000	(b) $0 \cdot 000512$

4.6 Logarithmic Systems

Logarithmic systems are of two kinds :

(a) Natural Logarithm :

The mathematician John Napier (1550 –1617) of Scotland first published the book on logarithm in 1614 by taking e as its base. e is an irrational number, $e = 2\cdot 718.......$ Such logarithm is called Napierian logarithm or e based logarithm or natural logarithm. $\log_e x$ is also written in the form $\ln x$.

(b) Common Logarithm :

The mathematician Henry Briggs (1561 – 1630) of England prepared log table in 1624 by taking 10 as the base. Such logarithm is called Briggs logarithm or 10 based logarithm or practical logarithm.

N.B. : If there is no mention of base, e in the case of expression (algebraic) and 10, in the case of number are considered the base. In log table 10 is taken as the base.

4.7 Characteristic and Mantissa of Common Logarithm

(a) Characteristics :

Let a number N be expressed in scientific form as $N = a \times 10^n$, where $N > 0, 1 \leq a < 10$ and $n \in Z$.

Taking log of both sides with base 10,

$$\log_{10} N = \log_{10}(a \times 10^n)$$

$$= \log_{10} a + \log_{10} 10^n = \log_{10} a + n\log_{10} 10$$

$$\therefore \log_{10} N = n + \log_{10} a \quad [\because \log_{10} 10 = 1]$$

$$\therefore \log_{10} N = n + \log_{10} a$$

Suppressing the base 10, we have,

$$\log N = n + \log a$$

n is called the characteristic of $\log N$.

We observe : Table-1

N	Form $a \times 10^m$ of N	Exponent	Number of digits on the left of the decimal point	Characteristic
6237	$6\cdot 237 \times 10^3$	3	4	$4-1=3$
$623\cdot 7$	$6\cdot 237 \times 10^2$	2	3	$3-1=2$
$62\cdot 37$	$6\cdot 237 \times 10^1$	1	2	$2-1=1$
$6\cdot 237$	$6\cdot 237 \times 10^0$	0	1	$1-1=0$
$0\cdot 6237$	$6\cdot 237 \times 10^{-1}$	-1	0	$0-1=-1$

We observe : Table-2

N	Form $a\times10^m$ of N	Exponent	Number of zoroes between decimal point and its next first significant digit	Characteristic
$0\cdot6237$	$6\cdot237\times10^{-1}$	-1	0	$-(0+1)=-1$
$0\cdot06237$	$6\cdot237\times10^{-2}$	-2	1	$-(1+1)=-2$
$0\cdot006237$	$6\cdot237\times10^{-3}$	-3	2	$-(2+1)=-3$

We observe from table-1 :

As many digits are there in the integral part of a number, characteristic of log of the number will be 1 less than that number of digits and that will be positive.

We observe from table 2 :

If there is no integral part of a number, as many zeroes are there in between decimal point and its next first significant digit, the characteristic of log of the number will be 1 more than the number of zeroes and that will be negative.

N. B. 1. Characteristic may be either positive or negative, but the mantissa will always be positive.

N. B. 2. If any characteristic is negative, not placing '–' sign on the left of the characteristic, it is written by giving '–' (bar sign) over the characteristic. Such as, characteristic –3 will be written as $\bar{3}$. Otherwise, whole part of the log including mantissa will mean negative.

Example 12. Find the characteristics of log of the following numbers :

(i) 5570 (ii) $45\cdot70$ (iii) $0\cdot4305$ (iv) $0\cdot000435$

Solution : (i) $5570=5\cdot570\times1000=5\cdot570\times10^3$

$\therefore$ Characteristic of log of the number is 3.

Otherwise, number of digits in the number 5570 is 4.

$\therefore$ Characteristic of log of the number is $=4-1=3$

$\therefore$ Characteristic of log of the number is 3.

(ii) $45\cdot70=4\cdot570\times10^1$

$\therefore$ Characteristic of log of the number is 1.

Otherwise, there are 2 digits in the integral part (i.e. on left of decimal point) of the number.

$\therefore$ Characteristic of the log of the number is $=2-1=1$

$\therefore$ Characteristic of log of the number is 45·70 is 1.

(iii) $0\cdot4305=4\cdot305\times10^{-1}$

$\therefore$ Characteristic of log of the number is -1

Otherwise, there is no significant digit in the integral part (before the decimal point) of the number or there is zero digit.

$\therefore$ Characteristic of log of the number $=0-1=-1=\bar{1}$

Again, there is no zero in between decimal point and its next first significant digit of the number $0 \cdot 4305$, i.e. there is 0 zeroes.

$\therefore$ Characteristic of log of the number is $= -(0+1) = -1 = \bar{1}$

$\therefore$ Characteristic of log of the number $0 \cdot 4305$ is $\bar{1}$

(iv) $0 \cdot 000435 = 4 \cdot 35 \times 10^{-4}$

$\therefore$ Characteristic of log of the number is -4 or $\bar{4}$

Otherwise, there are 3 zeroes in between decimal point and its next 1st significant digit.

$\therefore$ Characteristic of log of the number is $= -(3+1) = -4 = \bar{4}$

$\therefore$ Characteristic of log of the number is $0 \cdot 000435$ is $\bar{4}$

(b) Mantissa :

Mantissa of the Common Logarithm of any number is a non-negative number less than 1. It is mainly an irrational number. But the value of mantissa can be determined upto a certain places of decimal.

Mantissa of the log of a number can be found from log table. Again, it can also be found by calculator. We shall find the mantissa of the log of any number in 2nd method, that is by calculator.

Determination of common logarithm with the help of calculator :

Example 13. Find the characteristic and mantissa of $\log 2717$:

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{2717}$ $\boxed{=}$ $3 \cdot 43408$

$\therefore$ Characteristic of $\log 2717$ is 3 and mantissa is $\cdot 43408$

Example 14. Find the characteristic and mantissa of $\log\ 43 \cdot 517$.

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{43 \cdot 517}$ $\boxed{=}$ $1 \cdot 63866$

$\therefore$ Characteristic of log $43 \cdot 517$ is 1 and mantissa is $\cdot 63866$

Example 15. What are the characteristic and mantissa of the log of $0 \cdot 00836$?

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{0 \cdot 00836}$ $\boxed{=}$ $\bar{3} \cdot 92221 = \bar{3} \cdot 92221$

$\therefore$ Characteristic of log $0 \cdot 00836$ is -3 or $\bar{3}$ and mantissa is $\cdot 92221$

Example 16. Find $\log_e 10$

Solution : $\log_e 10 = \dfrac{1}{\log_{10} e} = \dfrac{1}{\log_{10}\ 2 \cdot 71828}$ [taking the value of e upto five decimal places]

$= \dfrac{1}{0 \cdot 43429}$ [using calculator]

$= 2 \cdot 30259$ (approx).

Alternative : We use the calculator :

AC	ln	10	=	2·30259 (approx).

Activity : Find the logarithm of the following numbers (each with the base 10 and e) by using calculator : (i) 2550 (ii) 52 ·143 (iii) 0·4145 (iv) 0·0742

Exercise 4.3

1. On what condition $a^0 = 1$?
   a. $a = 0$ b. $a \neq 0$ c. $a > 0$ d. $a \neq 1$
2. Which one of the following is the value of $\sqrt[3]{5} \cdot \sqrt[3]{5}$?
   a. $\sqrt[6]{5}$ b. $\left(\sqrt[3]{5}\right)^3$ c. $\left(\sqrt{5}\right)^3$ d. $\sqrt[3]{25}$
3. On what exact condition $\log_a a = 1$?
   a. $a > 0$ b. $a \neq 1$ c. $a > 0, a \neq 1$ d. $a \neq 0, a > 1$
4. If $\log_x 4 = 2$, what is the value of x ?
   a. 2 b. ± 2 c. 4 d. 10
5. What is the condition for which a number is to be written in the form $a \times 10^n$?
   a. $1 < a < 10$ b. $1 \leq a \leq 10$ c. $1 \leq a < 10$ d. $1 < a \leq 10$
6. Observe the following information :
   i. $\log_a (m)^p = p\log_a m$
   ii. $2^4 = 16$ and $log_2 16 = 4$ are synonymous.
   iii. $\log_a (m+n) = \log_a m + \log_a n$

Which of the above information are correct ?
   a. *i* and *ii* b. *ii* and *iii* c. *i* and *iii* d. *i*, *ii* and *iii*
7. What is the characteristic of the common log of $0 \cdot 0035$?
   a. 3 b. 1 c. $\overline{2}$ d. $\overline{3}$
8. Considering the number $0 \cdot 0225$, answer the following questions :
   (1) Which one of the following is of the form a^n of the number ?
   a. $(2 \cdot 5)^2$ b. $(\cdot 015)^2$ c. $(1 \cdot 5)^2$ d. $(\cdot 15)^2$
   (2) Which one of the following is the scientific form of the number ?
   a. 225×10^{-4} b. $22 \cdot 5 \times 10^{-3}$ c. $2 \cdot 25 \times 10^{-2}$ d. $\cdot 225 \times 10^{-1}$
   (3) What is the characteristic of the common log of the number ?
   a. $\overline{2}$ b. $\overline{1}$ c. 0 d. 2

9. Express into scientific form :
(a) 6530 (b) $60 \cdot 831$ (c) $0 \cdot 000245$ (d) 37500000
(e) $0 \cdot 00000014$
10. Express in the form of ordinary decimals :
(a) 10^5 (b) 10^{-5} (c) $2 \cdot 53 \times 10^4$ (d) $9 \cdot 813 \times 10^{-3}$
(e) $3 \cdot 12 \times 10^{-5}$
11. Find the characteristic of common logarithm of the following numbers (without using calculator) :
(a) 4820 (b) $72 \cdot 245$ (c) $1 \cdot 734$ (d) $0 \cdot 045$
(e) $0 \cdot 000036$
12. Find the characteristic and mantissa of the common logarithm of the following numbers by using calculator :
(a) 27 (b) $63 \cdot 147$ (c) $1 \cdot 405$ (d) $0 \cdot 0456$
(e) $0 \cdot 000673$
13. Find the common logarithm of the product/quotient (approximate value upto five decimal places) :
(a) $5 \cdot 34 \times 8 \cdot 7$ (b) $0 \cdot 79 \times 0 \cdot 56$ (c) $22 \cdot 2642 \div 3 \cdot 42$
(d) $0 \cdot 19926 \div 32 \cdot 4$
14. If $\log 2 = 0 \cdot 30103, \log 3 = 0 \cdot 47712$ and $\log 7 = 0 \cdot 84510$, find the value of the following expressions :
(a) $\log 9$ (b) $\log 28$ (c) $\log 42$
15. Given, $x = 1000$ and $y = 0 \cdot 0625$
a. Express x in the form $a^n b^n$, where a and b are prime numbers.
b. Express the product of x and y in scientific form.
c. Find the characteristic and mantissa of the common logarithm of xy.

Chapter five

Equations with One Variable

We have known in the previous class what equation is and learnt its usage. We have learnt the solution of simple equations with one variable and acquired knowledge thoroughly about the solution of simple equations by forming equations from real life problems. In this chapter, linear and quadratic equations and Identities have been discussed and their usages have been shown to solve the real life problems.

At the end of the chapter, the students will be able to –

- Explain the conception of variable
- Explain the difference between equation and identity
- Solve the linear equations
- Solve by forming linear equations based on real life problems
- Solve the quadratic equations and find the solution sets
- Form the quadratic equations based on real life problems and solve.

5·1 Variables

We know, $x+3=5$ is an equation. To solve it, we find the value of the unknown quantity x. Here the unknown quantity x is a variable. Again, to solve the equation $x+a=5$, we find the value of x, not the value of a. Here, x is assumed as variable and a as constant. In this case, we shall get the values of x in terms of a. But if we determine the value of a, we shall write $a=5-x$; that is, the value of a will be obtained in terms of x. Here a is considered a variable and x a constant. But if no direction is given, conventionally x is considered a variable. Generally, the small letters x, y, z, the ending part of English alphabet are taken as variables and a, b, c, the starting part of the alphabet are taken as constants.

The equation, which contains only one variable, is called a linear equation with one variable. Such as, in the equation $x+3=5$, there is only one variable x. So this is the linear equation with one variable.

We know what the set is. If a set $S=\{x : x \in R, 1 \leq x \leq 10\}$, x may be any real number from 1 to 10. Here, x is a variable. So, we can say that when a letter symbol means the element of a set, it is called variable.

Degree of an equation : The highest degree of a variable in any equation is called the degree of the equation. Degree of each of the equations $x+1=5$, $2x-1=x+5$, $y+7=2y-3$ is 1 ; these are linear equations with one variable.

Again, the degree of each of the equations $x^2+5x+6=0$, $y^2-y=12$, $4x^2-2x=3-6x$ is 2 ; these are quadratic equations with one variable. The equation $2x^3-x^2-4x+4=0$ is the equation of degree 3 with one variable.

5·2 Equation and Identity

Equation : There are two polynomials on two sides of the equal sign of an equation, or there may be zero on one side (mainly on right hand side). Degree of the variable of the polynomials on two sides may not be equal. Solving an equation, we get the number of values of the variable equal to the highest degree of that variable. This value or these values are called the roots of the equation. The equation will be satisfied by the root or roots. In the case of more than one root, these may be equal or unequal. Such as, roots of $x^2-5x+6=0$ are 2 an 3. Again, though the value of x in the equations $(x-3)^2=0$ is 3, the roots of the equation are 3, 3.

Identity` : There are two polynomials of same (equal) degree on two sides of equal sign. Identity will be satisfied by more values than the number of highest degree of the variable. There is no difference between the two sides of equal sign ; that is why, it is called identity. Such as, $(x+1)^2-(x-1)^2=4x$ is an identity ; it will be satisfied for all values of x. So this equation is an identity. Each algebraic formula is an identity. Such as, $(a+b)^2=a^2+2ab+b^2$, $(a-b)^2=a^2-2ab+b^2$, $a^2-b^2=(a+b)(a-b)$, $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ etc. are identities.

All equations are not identities, In identity '$\equiv$' sign is used instead of equal (=) sign. But as all identities are equations, in the case of identity also, generally the equal sign is used. Distinctions between equation and identity are given below :

Equation	Identity
1. Two polynomials may exist on both sides of equal sign, or there may be zero on one side.	1. Two polynomials exist on two sides.
2. Degree of the polynomials on both sides may be unequal.	2. Degree of the polynomials on both sides is equal.
3. The equality is true for one or more values of the variable.	3. Generally, the equality is true for all values of the original set of the variable.
4. The number of values of the variable does not exceed the highest degree of the equation	4. Equality is true for infinite number of values of the variable.
5. All equations are not formulae.	5. All algebraic formulae are identities.

Activity :	1. What is the degree of and how many roots has each of the following equations ? $(i)\ 3x+1=5$ $\quad(ii)\ \frac{2y}{5}-\frac{y-1}{3}=\frac{3y}{2}$ 2. Write down three identities.

5·3 Solution of the equations of first degree

In case of solving equations, some rules are to be applied. If the rules are known, solution of equations becomes easier. The rules are as follows :

1. If the same number or quantity is added to both sides of an equation, two sides remain equal.
2. If the same number or quantity is subtracted from both sides of an equation, two sides remain equal.
3. If both sides of an equation are multiplied by the same number or quantity, the two sides remain equal.
4. If both sides of an equation are divided by same nonzero number or quantity, the two sides remain equal.

The rules stated above may be expressed in terms of algebraic expressions as follows:

If $x=a$ and $c\neq 0$, (i) $x+c=a+c$ $\quad(ii)$ $x-c=a-c$ $\quad(iii)$ $xc=ac$ $\quad(iv)$ $\frac{x}{c}=\frac{a}{c}$

Besides, if a, b and c are three quantities, if $a=b+c$, $a-b=c$ and if $a+c=b$, $a=b-c$.

This law is known as transposition law and different equations can be solved by applying this law.

If the terms of an equation are in fractional form and if the degree of the variables in each numerator is 1 and the denominator in each term is constant, such equations are linear equations.

Example 1. Solve : $\frac{5x}{7}-\frac{4}{5}=\frac{x}{5}-\frac{2}{7}$

Solution : $\frac{5x}{7}-\frac{4}{5}=\frac{x}{5}-\frac{2}{7}$ or, $\frac{5x}{7}-\frac{x}{5}=\frac{4}{5}-\frac{2}{7}$ [by transposition]

or, $\frac{25x-7x}{35}=\frac{28-10}{35}$ or, $\frac{18x}{35}=\frac{18}{35}$

or, $18x=18$

or, $x=1$

$\therefore$ Solution is $x=1$.

Now, we shall solve such equations which are in quadratic form. These equations are transformed into their equivalent equations by simplifications and lastly the equations is transformed into linear equation of the form $ax=b$. Again, even if there are variables in the denominator, they are also transformed into linear equation by simplification.

Example 2. Solve : $(y-1)(y+2)=(y+4)(y-2)$

Solution : $(y-1)(y+2)=(y+4)(y-2)$

or, $y^2-y+2y-2=y^2+4y-2y-8$

or, $y-2=2y-8$

or, $y-2y=-8+2$ [by transposition]

or, $-y=-6$

or, $y=6$

$\therefore$ Solution is $y=6$

Example 3. Solve and write the solution set : $\frac{6x+1}{15}-\frac{2x-4}{7x-1}=\frac{2x-1}{5}$

Solution : $\frac{6x+1}{15}-\frac{2x-4}{7x-1}=\frac{2x-1}{5}$

or, $\frac{6x+1}{15}-\frac{2x-1}{5}=\frac{2x-4}{7x-1}$ [by transposition]

or, $\frac{6x+1-6x+3}{15}=\frac{2x-4}{7x-1}$ or, $\frac{4}{15}=\frac{2x-4}{7x-1}$

or, $15(2x-4)=4(7x-1)$ [by cross-multiplication]

or, $30x-60=28x-4$

or, $30x-28x=60-4$ [by transposition]

or, $2x=56$, or $x=28$

$\therefore$ Solution is $x=28$

and solution set is $S=\{28\}$.

Example 4. Solve : $\frac{1}{x-3}+\frac{1}{x-4}=\frac{1}{x-2}+\frac{1}{x-5}$

Solution : $\frac{1`}{x-3}+\frac{1}{x-4}=\frac{1}{x-2}+\frac{1}{x-5}$

or, $\frac{x-4+x-3}{(x-3)(x-4)}=\frac{x-5+x-2}{(x-2)(x-5)}$ or, $\frac{2x-7}{x^2-7x+12}=\frac{2x-7}{x^2-7x+10}$

Values of the fractions of two sides are equal. Again, numerators of two sides are equal, but denominators are unequal. In this case, if only the value of the numerators is zero, two sides will be equal.

$\therefore$ $2x-7=0$ or, $2x=7$

or, $x=\frac{7}{2}$

$\therefore$ Solution is $x=\frac{7}{2}$

Example 5. Find the solution set : $\sqrt{2x-3}+5=2$

Solution : $\sqrt{2x-3}+5=2$

or, $\sqrt{2x-3}=2-5$ [by transposition]

or, $\left(\sqrt{2x-3}\right)^2=(-3)^2$ [squaring both sides]

or, $2x-3=9$

or, $2x=12$

or, $x=6$

Since there is the sign of square root, verification of the correctness is necessary.

Putting $x=6$ in the given equation, we get,

$\sqrt{2\times 6-3}+5=2$ or $\sqrt{9}+5=2$

or, $3+5=2$

or, $8=2$, which is impossible.

$\therefore$ The equation has no solution.

$\therefore$ Solution set is :S = ϕ

Alternative method :

$\sqrt{2x-3}+5=2$

or, $\sqrt{2x-3}=2-5$

or, $\sqrt{2x-3}=-3$

Square root of any real quantity cannot be negative.

$\therefore$ The equation has no solution.

$\therefore$ Solution set is :S = ϕ

Activities : 1. If $\left(\sqrt{5}+1\right)x+4=4\sqrt{5}$, show that, $x=6-2\sqrt{5}$
2. Solve and write the solution set : $\sqrt{4x-3}+5=2$

5·4 Usage of linear equations

In real life we have to solve different types of problems. In most cases of solving these problems mathematical knowledge, skill and logic are necessary. In real cases, in the application of mathematical knowledge and skill, as on one side the problems are solved smoothly, on the other side in daily life, solutions of the problems are obtained by mathematics. As a result, the students are interested in mathematics. Here different types of problems based on real life will be expressed by equations and they will be solved.

For determining the unknown quantity in solving the problems based on real life, variable is assumed instead of the unknown quantity and then equation is formed by the given conditions. Then by solving the equation, value of the variable, that is the unknown quantity is found.

Example 6. The digit of the units place of a number consisting of two digits is 2 more than the digit of its tens place. If the places of the digits are interchanged, the number thus formed will be less by 6 than twice the given number. Find the number.

Solution : Let the digit of tens place be x. Then the digit of units place will be $x+2$.

$\therefore$ the number is $10x+(x+2)$ or, $11x+2$.

Now, if the places of the digits are interchanged, the changed number will be $10(x+2)+x$ or $11x+20$

By the question, $11x+20=2(11x+2)-6$
or, $11x+20=22x+4-6$
or, $22x-11x=20+6-4$ [by transposition]
or, $11x=22$
or, $x=2$
$\therefore$ The number is $11x+2=11\times2+2=24$
$\therefore$ given number is 24.

Example 7. In a class if 4 students are seated in each bench, 3 benches remain vacant. But if 3 students are seated on each bench, 6 students are to remain standing. What is the number of students in that class ?

Solution : Let the number of students in the class be x.
Since, if 4 students are seated in a bench, 3 benches remain vacant, the number of benches of that class $=\frac{x}{4}+3$
Again, since, if 3 students are seated in each bench, 6 students are to remain standing, the number of benches of that class $=\frac{x-6}{3}$
Since the number of benches is fixed,
$\frac{x}{4}+3=\frac{x-6}{3}$ or, $\frac{x+12}{4}=\frac{x-6}{3}$
or, $4x-24=3x+36$, or, $4x-3x=36+24$
or, $x=60$
$\therefore$ number of students of the class is 60.

Example 8. Mr. Kabir, from his Tk. 56000, invested some money at the rate of profit 12% per annum and the rest of the money at the rate of profit 10% per annum. After one year he got the total profit of Tk. 6400. How much money did he invest at the rate of profit 12% ?

Solution : Let Mr. Kabir invest Tk. x at the rate of profit 12%.
$\therefore$ he invested Tk. $(56000-x)$ at the rate of profit 10%.
Now, profit of Tk. x in 1 year is Tk. $x\times\frac{12}{100}\times1$, or, Tk. $\frac{12x}{100}$
Again, profit of Tk. $(56000-x)$ in 1 year is Tk. $(56000-x)\times\frac{10}{100}$,
or, Tk. $\frac{10(56000-x)}{100}$
By the question, $\frac{12x}{100}+\frac{10(56000-x)}{100}=6400$
or, $12x+560000-10x=640000$
or, $2x=640000-560000$

or, $2x = 80000$
or, $x = 40000$
$\therefore$ Mr. Kabir invested Tk. 40000 at the rate of profit 12%.

> **Activity :** Solve by forming equations :
> 1. What is the number, if any same number is added to both numerator and denominator of the fraction $\frac{3}{5}$, the fraction will be $\frac{4}{5}$?
> 2. If the difference of the squares of two consecutive natural numbers is 151, find the two numbers.
> 3. If 120 coins of Tk. 1 and Tk. 2 together are Tk. 180, what is the number of coins of each kind ?

Exercise 5·1

Solve (1-10) :

1. $3(5x-3) = 2(x+2)$ 2. $\frac{ay}{b} - \frac{by}{a} = a^2 - b^2$ 3. $(z+1)(z-2) = (z-4)(z+2)$

4. $\frac{7x}{3} + \frac{3}{5} = \frac{2x}{5} - \frac{4}{3}$ 5. $\frac{4}{2x+1} + \frac{9}{3x+2} = \frac{25}{5x+4}$ 6. $\frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$

7. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{a+b}{x-a-b}$ 8. $\frac{x-a}{b} + \frac{x-b}{a} + \frac{x-3a-3b}{a+b} = 0$ 9. $\frac{x-a}{a^2-b^2} = \frac{x-b}{b^2-a^2}$

10. $(3+\sqrt{3})z + 2 = 5 + 3\sqrt{3}$.

Find the solution set (11 - 19) :

11. $2x(x+3) = 2x^2 + 12$ 12. $2x + \sqrt{2} = 3x - 4 - 3\sqrt{2}$ 13. $\frac{x+a}{x-b} = \frac{x+a}{x+c}$

14. $\frac{z-2}{z-1} = 2 - \frac{1}{z-1}$ 15. $\frac{1}{x} + \frac{1}{x+1} = \frac{2}{x-1}$ 16. $\frac{m}{m-x} + \frac{n}{n-x} = \frac{m+n}{m+n-x}$

17. $\frac{1}{x+2} + \frac{1}{x+5} = \frac{1}{x+4} + \frac{1}{x+3}$ 18. $\frac{2t-6}{9} + \frac{15-2t}{12-5t} = \frac{4t-15}{18}$

19. $\frac{x+2b^2+c^2}{a+b} + \frac{x+2c^2+a^2}{b+c} + \frac{x+2a^2+b^2}{c+a} = 0$

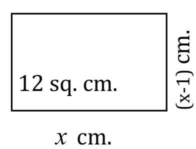
Solve by forming equations (20 - 27) :

20. A number is $\frac{2}{5}$ times of another number. If the sum of the numbers is 98, find the two numbers.
21. Difference of num. and denom. of a proper fraction is 1. If 2 is subtracted from numerator and 2 is added to denominator of the fraction, it will be equal to $\frac{1}{6}$. Find the fraction.
22. Sum of the digits of a number consisting of two digits is 9. If the number obtained by interchanging the places of the digits is less by 45 than the given number, what is the number ?
23. The digit of the units place of a number consisting of two digits is twice the digit of the tens place. Show that, the number is seven times the sum of the digits.
24. A petty merchant by investing Tk. 5600 got the profit 5% on some of the money and profit of 4% on the rest of the money. On how much money did he get the profit of 5% ?
25. Number of passengers in a launch is 47 ; the fare per head for the cabin is twice that for the deck. The fare per head for the deck is Tk. 30. If the total fare collected is Tk. 1680, what is the number of passengers in the cabin ?
26. 120 coins of twenty five paisa and fifty paisa together is Tk. 35. What is the number of coins of each kind ?
27. A car passed over some distance at the speed of 60 km per hour and passed over the rest of the distance at the speed of 40 km per hour. The car passed over the total distance of 240 km in 5 hours. How far did the car pass over at the speed of 60 km per hour ?

5·5 Quadratic equation with one variable

Equations of the form $ax^2 + bx + c = 0$ [where a, b, c are constants and $a \neq 0$] is called the quadratic equation with one variable. Left hand side of a quadratic equation is a polynomial of second degree, right hand side is generally taken to be zero.

Length and breadth of a rectangular region of area 12 square cm. are respectively x cm. and $(x-1)$ cm.



$\therefore$ area of the rectangular region is = $x(x-1)$ square cm.

By the question, $x(x-1) = 12$, or $x^2 - x - 12 = 0$.

x is the variable in the equation and highest power of x is 2.

Such equation is a quadratic equation. The equation, which has the highest degree 2 of the variable, is called the quadratic equation.

In class VIII, we have factorized the quadratic expressions with one variable of the forms $x^2 + px + q$ and $ax^2 + bx + c$. Here, we shall solve the equations of the forms $x^2 + px + q = 0$ and $ax^2 + bx + c = 0$ by factorizing the left hand side and by finding the value of the variable.

An important law of real numbers is applied to the method of factorization. The law is as follows :

If the product of two quantities is equal to zero, either ony one of the quantities or both quantities will be zero. That is, if the product of two quantities a and b i.e., $ab = 0$, $a = 0$ or, $b = 0$, or, both $a = 0$ and $b = 0$.

Example 9. Solve : $(x+2)(x-3)=0$

Solution : $(x+2)(x-3)=0$

$\therefore$ $x+2=0$, or, $x-3=0$

If $x+2=0$, $x=-2$

Again, if $x-3=0$, $x=3$

$\therefore$ solution is $x=-2$ or, 3.

Example 10. Find the solution set : $y^2=\sqrt{3}y$

Solution : $y^2=\sqrt{3}y$

or, $y^2-\sqrt{3}y=0$ [By transposition, right hand side has been done zero]

or, $y(y-\sqrt{3})=0$

$\therefore$ $y=0$, or $y-\sqrt{3}=0$

If $y-\sqrt{3}=0$, $y=\sqrt{3}$

$\therefore$ Solution set is $\{0,\sqrt{3}\}$.

Example 11. Solve and write the solution set : $x-4=\dfrac{x-4}{x}, x\neq 0$.

Solution : $x-4=\dfrac{x-4}{x}$

or, $x(x-4)=x-4$ [by cross-multiplication]

or, $x(x-4)-(x-4)=0$ [by transposition]

or, $(x-4)(x-1)=0$

$\therefore$ $x-4=0,$ or, $x-1=0$

If $x-4=0$, $x=4$

Again, if $x-1=0$, $x=1$

$\therefore$ Solution is : $x=1$ or, 4

and the solution set is $\{1, 4\}$.

Example 12. Solve : $\left(\dfrac{x+a}{x-a}\right)^2-5\left(\dfrac{x+a}{x-a}\right)+6=0$

Solution : $\left(\dfrac{x+a}{x-a}\right)^2-5\left(\dfrac{x+a}{x-a}\right)+6=0$..........(1)

Let, $\dfrac{x+a}{x-a}=y$

Then from (1), we get, $y^2-5y+6=0$

or, $y^2-2y-3y+6=0$

or, $y(y-2)-3(y-2)=0$

or, $(y-2)(y-3)=0$

$\therefore$ $y-2=0$, or, $y-3=0$

If $y-2=0$, $y=2$

If $y-3=0$, $y=3$

Now, when $y=2$,

$\frac{x+a}{x-a}=\frac{2}{1}$ [putting the value of y]

or, $\frac{x+a+x-a}{x+a-x+a}=\frac{2+1}{2-1}$ [by componendo and dividendo]

or, $\frac{2x}{2a}=\frac{3}{1}$

or, $x=3a$

Again, when $y=3$, $\frac{x+a}{x-a}=\frac{3}{1}$

or, $\frac{x+a+x-a}{x+a-x+a}=\frac{3+1}{3-1}$

or, $\frac{2x}{2a}=\frac{4}{2}$

or, $\frac{x}{a}=\frac{2}{1}$

or, $x=2a$

$\therefore$ Solution is : $x=3a$, or, $2a$

Activity :

1. Comparing the equation $x^2-1=0$ with the equation $ax^2+bx+c=0$, write down the values of a, b, c.
2. What is the degree of the equation $(x-1)^2=0$? How many roots has the equation and what are they?

5·6 Usage of quadratic equations

Many problems of our daily life can be solved easily by forming linear and quadratic equations. Here, the formation of quadratic equations from the given conditions based on real life problems and techniques for solving them are discussed.

Example 13. Denominator of a proper fraction is 4 more than the numerator. If the fraction is squared, its denominator will be 40 more than the numerator. Find the fraction.

Solution : Let the fraction be $\frac{x}{x+4}$.

Square of the fraction $= \left(\frac{x}{x+4}\right)^2 = \frac{x^2}{(x+4)^2} = \frac{x^2}{x^2+8x+16}$

Here, numerator $= x^2$ and denominator $= x^2+8x+16$.

By the question, $x^2+8x+16 = x^2+40$

or, $8x+16 = 40$

or, $8x = 40-16$

or, $8x = 24$

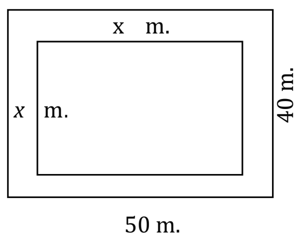
or, $x = 3$

$\therefore\ x+4 = 3+4 = 7$

$\therefore\ \frac{x}{x+4} = \frac{3}{3+4} = \frac{3}{7}$

$\therefore$ the fraction is $\frac{3}{7}$

Example 14. A rectangular garden with length 50 metre and breadth 40 metre has of equal width all around the inside of the garden. If the area of the garden except the path is 1200 square metre, how much is the path wide in metre ?



Solution : Let the path be x metre wide.

Without the path,

Length of the garden = $(50-2x)$ metre and

its breadth = $(40-2x)$ metre

$\therefore$ Without the path, area of the garden $= (50-2x)\times(40-2x)$ square metre.

By the question, $(50-2x)(40-2x) = 1200$

or, $2000-80x-100x+4x^2 = 1200$

or, $4x^2-180x+800 = 0$

or, $x^2-45x+200 = 0$ [dividing by 4]

or, $x^2-5x-40x+200 = 0$

or, $x(x-5)-40(x-5) = 0$

or, $(x-5)(x-40) = 0$

$\therefore\ x-5=0$, or $x-40=0$

If $x-5=0$, $x=5$

If $x-40=0$, $x=40$

But the breadth of the path will be less than 40 metre from the breadth of the garden.

$\therefore\ x \neq 40$; $\therefore\ x = 5$

$\therefore$ the path is 5 metres wide.

Example 15. Shahik bought some pens at Tk. 240. If he would get one more pen in that money, average cost of each pen would be less by Tk. 1. How many pens did he buy ?

Solution : Let, Shahik bought x pens in total by Tk. 240. Then each pen costs Tk. $\frac{240}{x}$. If he would get one more pen, that is, he would get $(x+1)$ pens, the cost of each pen would be Tk. $\frac{240}{x+1}$.

By the question, , $\frac{240}{x+1}=\frac{240}{x}-1$, or $\frac{240}{x+1}=\frac{240-x}{x}$

or, $240x=(x+1)(240-x)$ [by cross-multiplication]

or, $240x=240x+240-x^2-x$

or, $x^2+x-240=0$ [by transposition]

or, $x^2+16x-15x-240=0$

or, $x(x+16)-15(x+16)=0$

or, $(x+16)(x-15)=0$

$\therefore x+16=0$, or, $x-15=0$

If $x+16=0$, $x=-16$

Again, if $x-15=0$, $x=15$

But the number of pen x, cannot be negative.

$\therefore\ x\neq-16$; $\therefore\ x=15$

$\therefore$ Shahik bought 15 pens.

Activity : Solve by forming equations :

1. If a natural number is added to its square, the sum will be equal to nine times of exactly its next natural number. What is the number ?
2. Length of a perpendicular drawn from the centre of a circle of radius 10 cm. to a chord is less by 2 cm. than the semi-chord. Find the length of the chord by drawing a probable picture.

Example 16. In an examination of class IX of a school, total marks of x students obtained in mathematics is 1950. If at the same examination, marks of a new student in mathematics is 34 and it is added to the former total marks, the average of the marks become less by 1.

a. Write down the average of the obtained marks of all students including the new student and separately x students in terms of x.

b. By forming equation from the given conditions, show that, $x^2+35x-1950=0$

c. By finding the value of x, find the average of the marks in the two cases.

Solution : a. Average of the marks obtained by x students = $\frac{1950}{x}$

Average of the marks obtained by $(x+1)$ students including the new student = $\frac{[illegible] + 34}{x+1} = \frac{[illegible]}{x+1}$

b. By the question, $\frac{[illegible]}{x} = \frac{[illegible]}{x+1} + 1$

or, $\frac{[illegible]}{x} - \frac{[illegible]}{x+1} = 1$

or, $\frac{[illegible]\ x + [illegible] - [illegible]\ x}{x(x+1)} = 1$

or, $x^2 + x = [illegible]\ x - [illegible]\ x + [illegible]$ [by cross-multiplication]

or, $x^2 + x = [illegible] - 34x$

$\therefore$ $x^2 + 35x - [illegible] = 0$ [showed]

c. $x^2 + 35x - [illegible] = 0$

or, $x^2 + [illegible]\ x - 30x - [illegible] = 0$

or, $x(x + [illegible]) - 30(x + [illegible]) = 0$

or, $(x + [illegible])(x - 30) = 0$

$\therefore x + [illegible] = 0$, or, $x - 30 = 0$

If $x + [illegible] = 0$, $x = -[illegible]$

Again, if $x - 30 = 0$, $x = 30$

Since the number of students, i.e., x cannot be negative, $x \neq -[illegible]$

$\therefore$ $x = 30$.

$\therefore$ in the first case, average = $\frac{[illegible]}{30} = [illegible]$

and in the second case, average = $\frac{[illegible]}{31} = [illegible]$.

Exercise 5·2

1. Assuming x as the variable in the equation $a^2x + b = 0$, which one of the following is the degree of the equation ?

   a. 3 b. 2 c. 1 d. 0

2. Which one of the following is an identity ?

   a. $(x+1)^2 + (x-1)^2 = 4x$ b. $(x+1)^2 + (x-1)^2 = 2(x^2+1)$

   c. $(a+b)^2 - (a-b)^2 = 2ab$ d. $(a-b)^2 = a^2 + 2ab + b^2$

3. How many roots are there in the equation $(x-4)^2 = 0$?

   a. 1 b. 2 c. 3 d. 4

4. Which one of the following are the two roots of the equation $x^2 - x - [illegible] = 0$?

5. What is the coefficient of x in the equation $3x^2 - x + 5 = 0$?

a. 3 b. 2 c. 1 d. -1

6. Observe the following equations :

$i.\ 2x + 3 = 9 \qquad ii.\ \frac{x}{2} - 2 = -1 \qquad iii.\ 2x + 1 = 5$

Which are of the above equations equivalent ?

a. i and ii b. ii and iii c. i and iii d. i, ii and iii

7. Which one of the following is the solution set of the equation $x^2 - (a+b)x + ab = 0$?

a. $\{a, b\}$ b. $\{a, -b\}$ c. $\{-a, b\}$ d. $\{-a, -b\}$

8. The digit of the tens place of a number consisting of two digits is twice the digit of the units place. In respect of the information, answer the following questions :

(1) If the digit of the units place is x, what is the number ?

a. $2x$ b. $3x$ c. $12x$ d. $21x$

(2) If the places of the digits are interchanged, what will be the number ?

a. $3x$ b. $4x$ c. $12x$ d. $21x$

(3) If $x = 2$, what will be the difference between the original number and the number by interchanging their places?

a. 18 b. 20 c. 34 d. 36

Solve (9–18) :

9. $(x+2)(x-\sqrt{3}) = 0$ 10. $(\sqrt{2}x+3)(\sqrt{3}x-2) = 0$ 11. $y(y-5) = 6$

12. $(y+5)(y-5) = 24$ 13. $2(z^2-9) + 9z = 0$ 14. $\frac{3}{2z+1} + \frac{4}{5z-1} = 2$

15. $\frac{4}{\sqrt{10x-4}} + \sqrt{10x-4} = 5$ 16. $\frac{x-2}{x+2} + \frac{6(x-2)}{x-6} = 1$ 17. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}$

18. $\frac{x-a}{x-b} + \frac{x-b}{x-a} = \frac{a}{b} + \frac{b}{a}$

Find the solution set (19–25):

19. $\frac{3}{x} + \frac{4}{x+1} = 2$ 20. $\frac{x+7}{x+1} + \frac{2x+6}{2x+1} = 5$ 21. $\frac{1}{x} + \frac{1}{a} + \frac{1}{b} = \frac{1}{x+a+b}$

22. $\frac{ax+b}{a+bx} = \frac{cx+d}{c+dx}$ 23. $x + \frac{1}{x} = 2$ 24. $2x^2 - 4ax = 0$

25. $\dfrac{(x+1)^3-(x-1)^3}{(x+1)^2-(x-1)^2}=2$

Solve by forming equations (26–31) :

26. Sum of the two digits of a number consisting of two digits is 15 and their product is 56, find the number.
27. Area of the floor of a rectangular room is 192 square metre. If the length of the floor is decreased by 4 metre and the breadth is increased by 4 metre, the area remains unchanged. Find the length and breadth of the floor.
28. Length of the hypotenuse of a right angled triangle is 15 cm. and the difference of the lengths of other two sides is 3 cm. Find the lengths of those two sides.
29. The base of a triangle is 6 cm. more than twice its height. If the area of the triangle is 810 square cm., what is its height ?
30. As many students are there in a class, each of them contributes equal to the number of class-mates of the class and thus total Tk. 420 was collected. What is the number of students in the class and how much did each student contribute ?
31. As many students are there in a class, each of them contributed 30 paisa more than the number of paisa equal to the number of students and thus total Tk. 70 was collected. What is the number of students in that class ?
32. Sum of the digits of a number consisting of two digits is 7. If the places of the digits are interchanged, the number so formed is 9 more than the given number.
    a. Write down the given number and the number obtained by interchanging their places in terms of variable x.
    b. Find the given number.
    c. If the digits of the original number indicate the length and breadth of a rectangular region in centimetre, find the length of its diagonal. Assuming the diagonal as the side of a square, find the length of the diagonal of the square.
33. The base and height of a right angle triangle are respectively $(x-1)$ cm. and x cm. and the length of the side of a square is equal to the height of the triangle. Again, the length of a rectangular region is $(x+3)$ cm. and its breadth is x cm.
    a. Show the information in only one picture.
    b. If the area of the triangular region is 10 square centimetre, what is its height?
    c. Find the successive ratio of the areas of the triangular, square and rectangular regions.

Chapter Six
Lines, Angles and Triangles

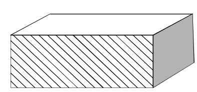
Geometry is an old branch of mathematics. The word ‘geometry’ comes from the Greek words ‘geo’, meaning the ‘earth’, and ‘metrein’, meaning ‘to measure’. So, the word ‘geometry’ means ‘the measurement of land.’ Geometry appears to have originated from the need for measuring land in the age of agricultural based civilization. However, now a days geometry is not only used for measuring lands, rather knowledge of geometry is now indispensible for solving many complicated mathematical problems. The practice of geometry is evident in relics of ancient civilization. According to the historians, concepts and ideas of geometry were applied to the survey of lands about four thousand years ago in ancient Egypt. Signs of application of geometry are visible in different practical works of ancient Egypt, Babylon, India, China and the Incas civilisation. In the Indian subcontinent there were extensive usages of gometry in the Indus Valley civilisation. The excavations at Harappa and Mohenjo-Daro show the evidence of that there was a well planned city. For example, the roads were parallel to each other and there was a developed underground drainage system. Besides the shape of houses shows that the town dwellers were skilled in mensuration. In Vedic period in the construction of altars (or *vedis*) definite geometrical shapes and areas were maintained. Usually these were constituted with triangles, quadrilaterals and trapeziums.

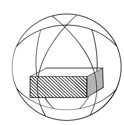
But geometry as a systematic discipline evolved in the age of Greek civilization. A Greek mathematician, Thales is credited with the first geometrical proof. He proved logically that a circle is bisected by its diameter. Thales’ pupil Pythagoras developed the theory of geometry to a great extent. About 300 BC Euclid, a Greek scholar, collected all the work and placed them in an orderly manner it in his famous treatise, ‘Elements’. ‘Elements’ completed in thirteen chapters is the foundation of modern geometry for generations to come. In this chapter, we shall discuss logical geometry in accordance with Euclid.

At the end of this chapter, the students will be able to

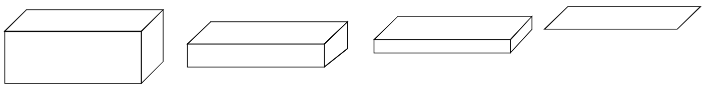
- Describe the basic postulates of plane geometry
- Prove the theorems related to triangles
- Apply the theorems and corollaries related to triangles to solve problems.

6·1 Concepts of space, plane, line and point





The space around us is limitless. It is occupied by different solids, small and large. By solids we mean the grains of sand, pin, pencil, paper, book, chair, table, brick, rock, house, mountain, the earth, planets and stars. The concepts of geometry springs from the study of space occupied by solids and the shape, size, location and properties of the space.



A solid occupies space which is spread in three directions. This spread in three directions denotes the three dimensions (length, breadth and height) of the solid. Hence every solid is three-dimensional. For example, a brick or a box has three dimensions (length, breadth and height). A sphere also has three dimensions, although the dimensions are not distinctly visible.

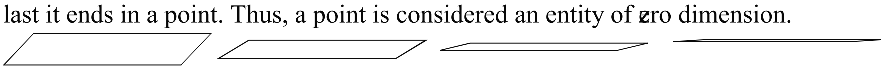
The boundary of a solid denotes a surface, that is, every solid is bounded by one or more surfaces. For example, the six faces of a box represent six surfaces. The upper face of a sphere is also a surface. But the surfaces of a box and of a sphere are different. The first one is plane while the second is one curved.

Two-dimensional surface : A surface is two dimensional; it has only length and breadth and is said to have no thickness. Keeping the two dimension of a box unchanged, if the third dimension is gradually reduced to zero, we are left with a face or boundary of the box. In this way, we can get the idea of surface from a solid.

When two surfaces intersect, a line is formed. For example, two faces of a box meet at one side in a line. This line is a straight line. Again, if a lemon is cut by a knife, a curved line is formed on the plane of intersection of curved surface of the lemon.

One-dimensional line : A line is one-dimensional; it has only length and no breadth or thickness. If the width of a face of the box is gradually receded to zero, we are left with only line of the boundary. In this way, we can get the idea of line the from the idea of surface.

The intersection of two lines produces a point. That is, the place of intersection of two lines is denoted by a point. If the two edges of a box meet at a point. A point has no length, breadth and thickness. If the length of a line is gradually reduced to zero at last it ends in a point. Thus, a point is considered an entity of zero dimension.



6·2 Euclid's Axioms and Postulates

The discussion above about surface, line and point do not lead to any definition – they are merely description. This description refers to height, breadth and length, neither of which has been defined. We only can represent them intuitively. The definitions of point, line and surface which Euclid mentioned in the beginning of the first volume of his Elements' are incomplete from modern point of view. A few of **Euclid's axioms** are given below:

1. A **point** is that which has no part.
2. A line has no end point.

3. A **line** has only length, but no breath and height.

4. A **straight line** is a line which lies evenly with the points on itself.

5. A **surface** is that which has length and breadth only.

6. The edges of a surface are lines.

7. A **plane surface** is a surface which lies evenly with the straight lines on itself.

It is observed that in this description, part, length, width, evenly etc have been accepted without any definition. It is assumed that we have primary ideas about them. The ideas of point, straight line and plane surface have been imparted on this assumption. As a matter of fact, in any mathematical discussion one or more elementary ideas have to be taken granted. Euclid called them axioms. Some of the axioms given by Euclid are:

(1) Things which are eqal to the same thing, are eqal to one another.

(2) If eqals are added to eqals, the wholes are eqal.

(3) If eqals are subtracted from eqals, the remainders are eqal.

(4) Things which coincide with one another, are eqal to one another.

(5) The whole is greater than the part.

In modern geometry, we take a point, a line and a plane as undefined terms and some of their properties are also admitted to be true. These admitted properties are called geometric postulates. These postulates are chosen in such a way that they are consistent with real conception. The five postulates of Euclid are:

Postulate 1: *A straight line may be drawn from any one point to any other point.*

Postulate 2: *A terminated line can be produced indefinitely,*

Postulate 3: *A circle can be drawn with any centre and any radius.*

Postulate 4: *All right angles are equal to one another.*

Postulate 5: *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called **propositions or theorems**. Euclid is his ‘elements’ proved a total of 465 propositions in a logical chain. This is the foundation of modern geometry.

Note that there are some incompletion in Euclid’s first postulate. The drawing of a uniqe straight line passing through two distinct points has been ignored.

Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as ‘self-evident truths’. However, it is not possible to prove them. So, these statements are accepted without any proof. Since the fifth postulate is related to parallel lines, it will be discussed later.

6.3 Plane Geometry

It has been mentioned earlier that point, straight line and plane are three fundamental concepts of geometry. Although it is not possible to define them properly, based on our real life experience we have ideas about them. As a concrete geometrical conception space is regarded as a set of points and straight lines and planes are considered the subsets of this universal set.

Postulate 1. Space is a set of all points and plane and straight lines are the subsets of this set. From this postulate we observe that each of plane and straight line is a set and points are its elements. However, in geometrical description the notation of sets is usually avoided. For example, a point included in a straight line or plane is expressed by the point lies on the straight line or plane' or the straight line or plane passes through the point'. Similarly if a straight line is the subset of a plane, it is expressed by such sentences as the straight line lies on the plane, or the plane passes through the straight line'.

It is accepted as properties of straight line and plane that,

Postulate 2. For two different points there exists one and only one straight line, on which both the points lie.

Postulate 3. For three points which are not collinear, there exists one and only one plane, on which all the three points lie.

Postulate 4. A straight line passing through two different points on a plane lie completely in the plane.

Postulate 5. (a) Space contains more than one plane

(b) In each plane more than one straight lines lie.

(c) The points on a straight line and the real numbers can be related in such a way that every point on the line corresponds to a unique real number and conversely every real number corresponds to a unique point of the line.

Remark: The postulates from 1 to 5 are called incidence postulates.

The concept of distance is also an elementary concept. It is assumed that,

Postulate 6 : (a) Each pair of points (P, Q) determines a unique real number which is known as the ***distance*** between point P and Q and is denoted by PQ.

(b) If P and Q are different points, the number PQ is positive. Otherwise, $PQ=0$.

(c) The distance between P and Q and that between Q and P are the same, i.e. $PQ=QP$.

According to postulate 5(c) one to one correspondence can be established between the set of points in every straight line and the set of real numbers. In this connection, it is admitted that,

Postulate 7 : One-to-one correspondence can be established between the set of points in a straight line and the set of real numbers such that, for any points *P* and *Q*, $PQ = |a - b|$ where, the one-to-one correspondence associates points *P* and *Q* to real numbers *a* and *b* respectively.

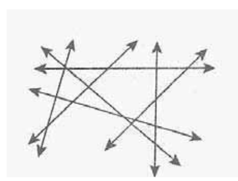
If the correspondence stated in this postulate is made, the line is said to have been reduced to a number line. If *P* corresponds to *a* in the number line, *P* is called the graph point of *P* and *a* the coordinates of *P*. To convert a straight line into a number line the coordinates of two points are taken as 0 and 1 respectively. Thus a unit distance and the positive direction are fixed in the straight line. For this, it is also admitted that,

Postulate 8: Any straight line *AB* can be converted into a number line such that the coordinate of *A* is 0 and that of *B* is positive.

Remark: Postulate 6 is known as distance postulate and Postulate 7 as ruler postulate and Postulate 8 as ruler placement postulate.

Geometrical figures are drawn to make geometrical description clear. The model of a point is drawn by a thin dot by tip of a pencil or pen on a paper. The model of a straight line is constructed by drawing a line along a ruler. The arrows at ends of a line indicate that the line is extended both ways indefinitely. By postulate 2, two different points *A* and *B* define a unique straight line on which the two points lie. This line is called *AB* or *BA* line. By postulate 5(c) every such straight line contains infinite number of points.

According to postulate 5(a) more than one plane exist. There is infinite number of straight lines in every such plane. *The branch of geometry that deals with points, lines and different geometrical entities related to them, is known as plane Geometry.* In this textbook, plane geometry is the matter of our discussion. Hence, whenever something is not mentioned in particular, we will assume that all discussed points, lines etc lie in a plane.



Proof of Mathematical statements

In any mathematical theory different statements related to the theory are logically established on the basis of some elementary concepts, definitions and postulates. Such statements are generally known as propositions. In order to prove correctness of statements some methods of logic are applied. The methods are:

(a) Method of induction
(b) Method of deduction

Proof by contradiction

Philosopher Aristotle first introduced this method of logical proof. The basis of this method is:

- A property can not be accepted and rejected at the same time.
- The same object can not possess opposite properties.
- One can not think of anything which is contradictory to itself.
- If an object attains some property, that object can not unattain that property at the same time.

6·4 Geometrical proof

In geometry, special importance is attached to some propositions which are taken, as theorems and used successively in establishing other propositions. In geometrical proof different statements are explained with the help of figures. But the proof must be logical.

In describing geometrical propositions general or particular enunciation is used. The general enunciation is the description independent of the figure and the particular enunciation is the description based on the figure. If the general enunciation of a proposition is given, subject matter of the proposition is specified through particular enunciation. For this, necessary figure is to be drawn.

Generally, in proving the geometrical theorem the following steps should be followed :

(1) General enunciation.

(2) Figure and particular enunciation.

(3) Description of the necessary constructions and

(4) Description of the logical steps of the proof.

If a proposition is proved directly from the conclusion of a theorem, it is called a corollary of that theorem. Besides, proof of various propositions, proposals for construction of different figures are considered. These are known as constructions. By drawing figures related to problems, it is necessary to narrate the description of construction and its logical truth.

Exercise 6.1

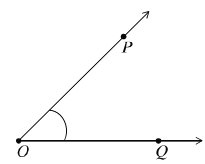
1. Give a concept of space, surface, line and point.
2. State Euclid's five postulates.
3. State five postulates of incidence.
4. State the distance postulate.
5. State the ruler postulate.
6. Explain the number line.

7 State the postulate of ruler placement.
8 Define intersecting straight line and parallel straight line.

Line, Ray and Line Segment

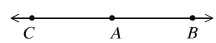
By postulates of plane geometry, every point of a straight line lies in a plane. Let AB be a line in a plane and C be a point on it. The point C is called internal to A and B if the points A, C and B are different points on a line and $AC + CB = AB$. The points A, C and B are also called collinear points. The set of points including A and B and all the internal points is known as the line segment AB. The points between A and B are called internal points.

Angles



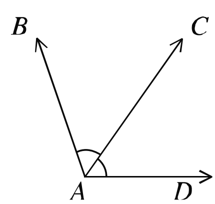
When two rays in a plane meet at a point, an angle is formed. The rays are known as the sides of the angle and the common point as vertex. In the figure, two rays OP and OQ make an angle $\angle POQ$ at their common point O. O is the vertex of the angle. The set of all points lying in the plane on the Q side of OP and P side of OQ is the known as the interior region of the $\angle POQ$. The set of all points not lying in the interior region or on any side of the angle is called exterior region of the angle.

Straight Angle



The angle made by two opposite rays at their common end point is a straight angle. In the adjacent figure, a ray AC is drawn from the end point A of the ray AB. Thus the rays AB and AC have formed an angle $\angle BAC$ at their common point A. $\angle BAC$ is a straight angle. The measurement of a right angle is 2 right angles or 180^0.

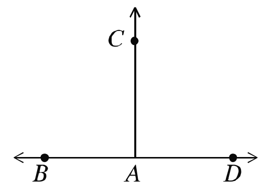
Adjacent Angle



If two angles in a plane have the same vertex, a common side and the angles lie on opposite sides of the common side, each of the two angles is said to be an adjacent angle of the other. In the adjacent figure, the angles $\angle BAC$ and $\angle CAD$ have the same vertex A, a common side AC and are on opposite sides of AC. $\angle BAC$ and $\angle CAD$ are adjacent angles.

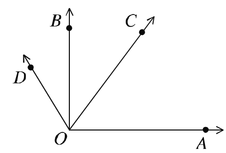
Perpendicular and Right Angle

The bisector of a straight angle is called perpendicular and each of th related adjacent angle a right angle. In the adjacent figure two angles $\angle BAC$ and $\angle CAD$ are produced at the point A of BD. The angles $\angle BAC$ and $\angle CAD$ are equal and lie on opposite sides of the common side AC. Each of the angles $\angle BAC$ and $\angle CAD$ is a right angle and the line segments BD and AC are mutually perpendicular.



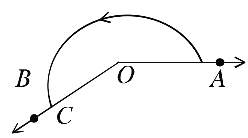
Acute and Obtuse Angles

An angle which is less than a right angle is called an acute angle and an angle greater than one right angle but less than two right angles is an obtuse angle. In the figure, $\angle AOC$ is an acute angle and $\angle AOD$ is an obtuse angle



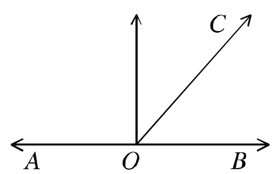
Reflex Angle

An angle which is greater than two right angles and less than four right angles is called a reflex angle. In the figure, $\angle AOC$ is a reflex angle.



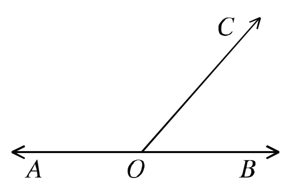
Complementary Angles

If the sum of two angles is one right angle, the two angles are called complementary angles. In the adjacent figure, $\angle AOB$ is a right angle. The ray OC is in the inner side of the angle and makes two angles $\angle AOC$ and $\angle COB$. So taking together the measurement of these two angles is one right angle. The angles $\angle AOC$ and $\angle COB$ are complementary angles.



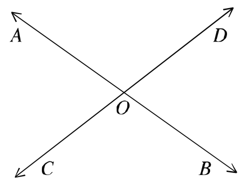
Supplementary Angles

If the sum of two angles is 2 right angles, two angles are called supplementary angles. The point O is an internal point of the line AB. OC is a ray which is different from the ray OA and ray OB. As a result two angles $\angle AOC$ and $\angle COB$ are formed. The measurement of this two angles is equal to the measurement of the straight angle $\angle AOB$ i.e., two right angles. The angles $\angle AOC$ and $\angle COB$ are supplementary angles.



Opposite Angles

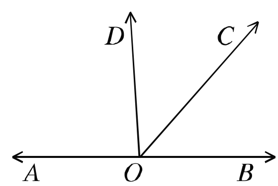
Two angles are said to be the opposite angles if the sides of one are the opposite rays of the other.



In the adjoining figure OA and OB are mutually opposite rays. So are the rays OC and OD. The angles $\angle AOC$ and $\angle BOD$ are a pair of opposite angles. Similarly, $\angle BOC$ and $\angle DOA$ are another pair of opposite angles. Therefore, two intersecting lines produce two pairs of opposite angles.

Theorem 1

The sum of the two adjacent angles which a ray makes with a straight line on its meeting point is equal to two right angles.



Let, the ray OC meets the straight line AB at O. As a result two adjacent angles $\angle AOC$ and $\angle COB$ are formed. Draw a perpendicular DO on AB.

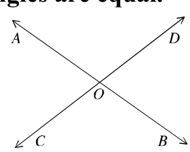
Sum of the adjacent two angles = $\angle AOC + \angle COB = \angle AOD + \angle DOC + \angle COB$

$= \angle AOD + \angle DOB$

= 2 right angles. [Proved]

Theorem 2

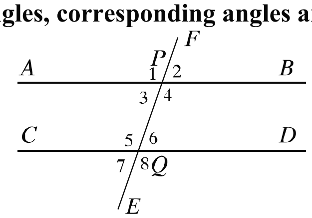
When two straight lines intersect, the vertically opposite angles are equal.



Let AB and CD be two straight lines, which intersect at O. As a result the angles $\angle AOC$, $\angle COB$, $\angle BOD$, $\angle AOD$ are formed at O. $\angle AOC$ = opposite $\angle BOD$ and $\angle COB$ = opposite $\angle AOD$.

6·4 Parallel lines

Alternate angles, corresponding angles and interior angles of the traversal



In the figure, two straight lines AB and CD are cut by a straight line EF at P and Q. The straight line EF is a traversal of AB and CD. The traversal has made eight angles $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5, \angle 6, \angle 7, \angle 8$ with the lines AB and CD. Among the angles

(a) $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$ are corresponding angles,

(b) $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$ are alternate angles,

(c) $\angle 4$, $\angle 6$ are interior angles on the right

(d) $\angle 3$, $\angle 5$ are interior angles on the left.

In a plane two straight lines may intersect or they are parallel. The lines intersect if there exists a point which is common to both lines. Otherwise, the lines are parallel. Note that two different straight lines may atmost have only one point in common.

The parallelism of two straight lines in a plane may be defined in three different ways:

(a) The two straight lines never intersect each other (even if extended to infinity)

(b) Every point on one line lies at equal smallest distance from the other.

(c) The corresponding angles made by a transversal of the pair of lines are equal.

According to definition (a) in a plane two straight lines are parallel, if they do not intersect each other. Two line segments taken as parts of the parallel lines are also parallel.

According to definition (b) the perpendicular distance of any point of one of the parallel lines from the other is always equal. Perpendicular distance is the length of the perpendicular from any point on one of the lines to the other. Conversely, if the perpendicular distances of two points on any of the lines to the other are equal, the lines are parallel. This perpendicular distance is known as the distance of the parallel lines.

The definition (c) is equivalent to the fifth postulate of Euclid. This definition is more useful in geometrical proof and constructions.

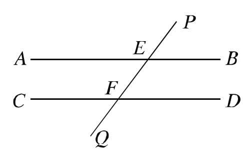
Observe that, through a point not on a line, a unique line parallel to it can be drawn.

Theorem 3

When a transversal cuts two parallel straight lines,

(a) the pair of alternate angles are equal.

(b) that pair of interior angles on the same side of the transversal are supplementary.



In the figure $AB \parallel CD$ and the transversal PQ intersects them at E and F respectively. Therefore,

(a) $\angle PEB$ = corresponding $\angle EFD$ [by definition]

(b) $\angle AEF$= alternate $\angle EFD$

(c) $\angle BEF + \angle EFD$ == 2 right angles.

Activity:

1. Using alternate definitions of parallel lines prove the theorems related to parallel straight lines.

Theorem 4

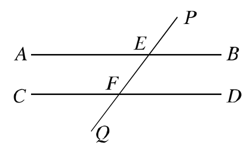
When a transversal cuts two straight lines, such that

(a) pairs of corresponding angles are equal, or

(b) pairs of alternate interior angles are equal, or

(c) pairs of interior angles on the same side of the transversal are or equal to, the sum of two right angles the lines are parallel.

In the figure the line *PQ* intersects the straight lines *AB* and *CD* at *E* and *F* respectively and



(a) $\angle PEB$ = alternate $\angle EFD$

or, (b) $\angle AEF$ = Corresponding $\angle EFD$

or, (c) $\angle BEF + \angle EFD$ = 2 right angles.

Therefore, the straight lines *AB* and *CD* are parallel.

Corollary 1. The lines which are parallel to a given line are parallel to each other.

Exercise 6.2

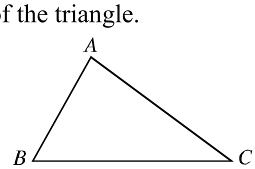
1. Define interior and exterior of an angle.
2. If there are three different points in a line, identify the angles in the figure.
3. Define adjacent angles and locate its sides.
4. Define with a figure of each: opposite angles, complementary angle, supplementary angle, right angle, acute and obtuse angle.

6.5 Triangles

A triangle is a figure closed by three line segments. The line segments are known as sides of the triangle. The point common to any pair of sides is the vertex. The sides form angles at the vertices. A triangle has three sides and three angles. Triangles are classified by sides into three types: equilateral, isosceles and scalene. By angles triangles are also classified into three types: acute angled, right angled and obtuse angled.

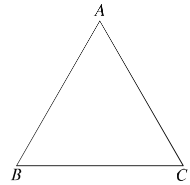
The sum of the lengths of three sides of the triangle is the perimeter. By triangle we also denote the region closed by the sides. The line segment drawn from a vertex to the mid-point of opposite side is known as the *median.* Again, the perpendicular distance from any vertex to the opposite side is the *height* of the triangle.

In the adjacent figure ABC is a triangle. A, B, C are three vertices. AB, BC, CA are three sides and $\angle BAC$, $\angle ABC$, $\angle BCA$ are three angles of the triangle. The sum of the measurement of AB, BC and CA is the perimeter of the triangle.



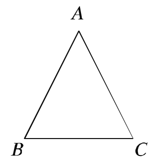
Equilateral Triangle

An equilateral triangle is a triangle of three equal sides. In the adjacent figure, triangle ABC is an equilateral triangle; because, $AB = BC = CA$ i.e., the lengths of three sides are equal.



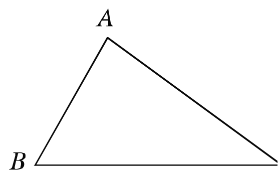
Isosceles Triangle

An isosceles triangle is triangle with two equal sides. In the adjacent figure triangle ABC is an isosceles triangle; because $AB = AC \neq BC$ i.e., the lengths of only two sides are equal.



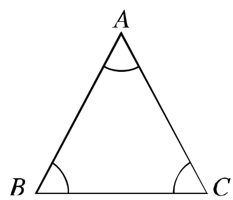
Scalene Triangle

Sides of scalene triangle are unequal. Triangle ABC is a scalene triangle, since the lengths of its sides AB, BC, CA are unequal.



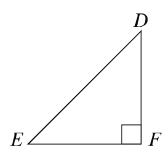
Acute Angled Triangle

A triangle having all the three angles acute is acute angled triangle. In the triangle ABC each of the angles $\angle BAC$, $\angle ABC$ and $\angle BCA$ is acute i.e., the measurement of any angle is less than 90°. So ΔABC is acute angled.



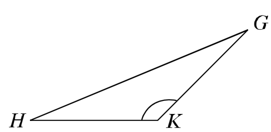
Right Angled Triangle

A triangle with one of the angles right is a right angled triangle. In the figure, the $\angle DFE$ is a right angle; each of the two other angles $\angle DEF$ and $\angle EDF$ are acute. The triangle ΔDEF is a right angled triangle.



Obtuse angled triangle

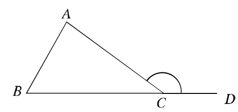
A triangle having an angle obtuse is an obtuse angled triangle. In the figure, the $\angle GKH$ is an obtuse angle; the two other angles $\angle GHK$ and $\angle HGK$ are acute. ΔGHK is an obtuse angled triangle.



9.3 Interior and Exterior Angles

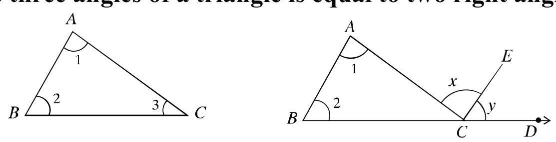
If a side of a triangle is produced, a new angle is formed. This angle is known as exterior angle. Except the angle adjacent to the exterior angle, the two others angles of the triangle are known as opposite interior angles.

In the adjacent figure, the side BC of ΔABC is produced to D. The angle $\angle ACD$ is an exterior angle of the triangle. $\angle ABC$, $\angle BAC$ and $\angle ACB$ are three interior angles. $\angle ACB$ is the adjacent interior angle of the exterior angle $\angle ACD$. Each of $\angle ABC$ and $\angle BAC$ is an opposite interior angle with respect to $\angle ACD$.



Theorem 5

The sum of the three angles of a triangle is equal to two right angles.



Let ABC be a triangle. In the triangle $\angle BAC + \angle ABC + \angle ACB = 2$ right angles.

Corollary 1: If a side of a triangle is produced then exterior angle so formed is equal to the sum of the two opposite interior angles.

Corollary 2: If a side of a triangle is produced, the exterior angle so formed is greater than each of the two interior opposite angles.

Corollary 3: The acute angles of a right angled triangle are complementary to each other.

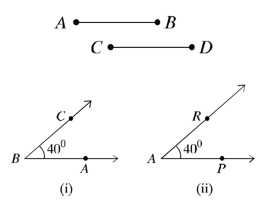
Activity :

1 Prove that if a side of a triangle is produced, the exterior angle so formed is greater than each of the two interior opposite angles.

Congruence of Sides and Angles

If two line segments have the same length, they are congruent. Conversely, if two line segments are congruent, they have the same length.

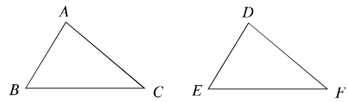
If the measurement of two angles is equal, the angles are congruent. Conversely, if two angles are congruent, their measurement is the same.



Congruence of Triangles

If a triangle when placed on another exactly covers the other, the triangles are congruent. The corresponding sides and angles of two congruent triangles are equal.

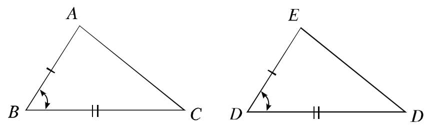
In the figure, ΔABC and ΔDEF are congruent. If two triangles ABC and DEF are congruent, by superposition of a copy of ABC on DEF we find that each covers the other completely. . Hence, the line segments as well angles are congruent. We would express this as $\Delta ABC \cong \Delta ABC$.



Theorem 6 (SAS criterion)

If two sides and the angle included between them of a triangle are equal to two corresponding sides and the angle included between them of another triangle, the triangles are congruent.

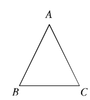
Let ABC and DEF be two triangles in which $AB = DE$, $AC = DF$ and the included $\angle BAC$ = the included $\angle EDF$. Then $\Delta ABC \cong \Delta DEF$.



Theorem 7

If two sides of a triangle are equal, the angles opposite the equal sides are also equal.

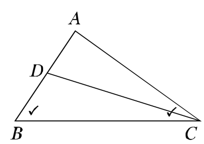
Suppose in the triangle ABC, $AB = AC$, then $\angle ABC = \angle ACB$.



Theorem 8

If two angles of a triangle are equal, the sides opposite the equal angles are also equal.

In the triangle ABC $\angle ABC = \angle ACB$. It is to be proved that $AB = AC$.



Proof

Steps	Justification
(1) If $AB = AC$ is not equal to AB, (i) $AB > AC$ or, (ii) $AB < AC$. Suppose, $AB > AC$. Cut from AB a part AD equal to AC. Now, the triangle ADC is an isosceles triangle. So, $\angle ADC = \angle ACD$	[The base angles of an isosceles triangles are equal]
In ΔDBC Exterior angle $\angle ADC > \angle ABC$ $\therefore \angle ACD > \angle ABC$ Therefore, $\angle ACB > \angle ABC$ But this is against the given condition.	[Exterior angle is greater than each of the interior opposite angles]

(2) Similarly, (ii) If $AB < AC$, it can be proved that $\angle ABC > \angle ACB$.

But this is also against the condition,

(3) So neither $AB > AC$ nor $AB < AC$

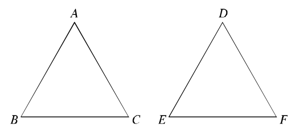
$\therefore\ AB = AC$ (Proved)

Theorem 9 (SSS criterion)

If the three sides of one triangle are equal to the three corresponding sides of another triangle, the triangles are congruent.

In ΔABC and ΔDEF, $AB = DE$, $AC = DF$ and $BC=EF$,

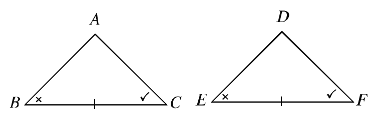
$\Delta ABC \cong \Delta ABC$.



Theorem 10 (ASA criterion)

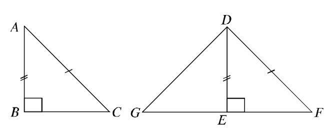
If two angles and the included side of a triangle are equal to two corresponding angles and the included side of another triangle, the triangles are congruent.

Let ABC and DEF be two triangles in which the $\angle B = \angle E$, $\angle C = \angle F$. and the side BC = the corresponding side EF, then the triangles are congruent, i.e. $\Delta ABC \cong \Delta DEF$.



Theorem 11 (HSA criterion)

If the hypotenuse and one side of a right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, the triangles are congruent.

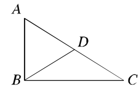


Let ABC and DEF be two right angled triangles, in which the hypotenuse AC= hypotenuse DF and $AB = DE$, then $\Delta ABC \cong \Delta DEF$. .

Theorem 12

If one side of a triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the lesser sides.

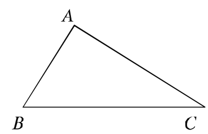
Let ABC be a triangle whose $AC > AB$, then $\angle ABC > \angle ACB$.



Theorem 13

If one angle of a triangle is greater than another, the side opposite the greater angle is greater than the side opposite the lesser.

Let ABC be a triangle in which $\angle ABC > \angle ACB$.
It is required to prove that, $AC > AB$.



Proof :

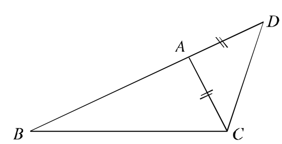
Steps	Justification
(1) If the side AC is not greater than AB, (i) $AC = AB$ or, (ii) $AC < AB$	
(i) if $AC=AB$ $\angle ABC = \angle ACB$ which is against the supposition, since by supposition $\angle ABC > \angle ACB$	[The base angles of isosceles triangle are equal]
(ii) Again if $AC < AB$, $\angle ABC < \angle ACB$ But this is also against the supposition.	[The angle opposite to smaller side is smaller]
(2) Therefore, the side AC is neither equal to nor less than AB. $\therefore AC > AB$ (Proved).	

There is a relation between the sum or the differece of the lengths of two sides and the length of the third side of a triangle.

Theorem 14

The sum of the lengths of any two sides of a triangle is greater than the third side.

Let ABC be a triangle. Then any two of its sides are together greater than the third side. Let BC to be the greatest side. Then $AB+AC > BC$.



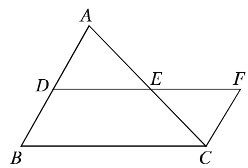
Corollary 1. The difference of the lengths of any two sides of a triangle is smaller than the third side.

Let ABC be a triangle. Then the difference of the lengths of any two of its sides is smaller than the length of third side i.e. $AB - AC < BC$.

Theorem 15

The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and in length it is half.

Let ABC be a triangle and D and E are respectively mid-points of the AB and AC. It is required to prove that $DE \parallel BC$ and $DE = \frac{1}{2} BC$



Construction: Join D and E and extend to F so that $EF = DE$.

Proof :

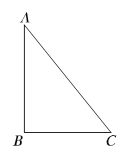
Steps	Justification
(1) Between ΔADE and ΔCEF, $AE = EC$	[given]
$DE = EF$	[by construction]
$\angle AED = \angle CEF$	[opposite angles]
$\Delta ADE \cong \Delta CEF$	[SAS theorem]
$\therefore \angle ADE = \angle EFC$ and $\angle DAE = \angle ECF$.	
$\therefore DF \parallel BC$ or, $DE \parallel BC$.	

(2) Again, $DF = BC$ or, $DE + EF = BC$ or, $DE + DE = BC$ or, $DE = \frac{1}{2} BC$

Theorem 16 (Pythagoras theorem)

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares of regions on the two other sides.

Let in the triangle ABC, $\angle ABC = 1$ right angle and AC is the hypotenuse.
Then $AC^2 = AB^2 + BC^2$,



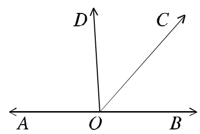
Exercise 6.3

1. The lengths of three sides of a triang le are given below. In which case it is possible to draw a triangle?
   (a) 5 cm, 6 cm and 7 cm (b) 3 cm, 4 cm and 7 cm
   (c) 5 cm, 7 cm and 14 cm (a) 2 cm, 4 cm and 8 cm
2. Consider the following information:
   (i) A right angled triangle is a triangle with each of three angles right angle.
   (ii) An acute angled triangle is a triangle with each of three angles acute.
   (iii) A triangle with all sides equal is an equilateral triangle.

Which one of the following is correct?

(a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii

Use the figure below to answer questions 3 and 4.



3. Which one is a right angle?

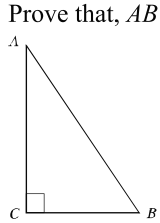
   (a) $\angle BOC$ (b) $\angle BOD$ (c) $\angle COD$ (d) $\angle AOD$

4. What is the angle complementary to BOC?

   (a) $\angle AOC$ (b) $\angle BOD$ (c) $\angle COD$ (d) $\angle AOD$

5. Prove that, the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.
6. Prove that, the three medians of an equilateral triangle are equal.
7. Prove that, the sum of any two exterior angles of a triangle is greater than two right angles.
8. D is a point inside a triangle ABC. Prove that, $AB + AC > BD + DC$.
9. If D is the middle point of the side BC of the triangle ABC, prove that, $AB + AC > 2AD$.
10. Prove that, the sum of the three medians of a triangle is less than its perimeter.
11. A is the vertex of an isosceles triangle ABC, and the side BA is produced to D such that $BA = AD$; prove that $\angle BCD$ = right angle.
12. The bisectors of the angles $\angle B$ and $\angle C$ of a triangle ABC intersect at O. Prove that $\angle BOC = 90° + \frac{1}{2} \angle A$.
13. If the sides AB and AC of a triangle ABC are produced and the bisector of the exterior angles formed at B and C meet at O, prove that, $\angle BOC = 90° - \frac{1}{2} \angle A$.
14. In the adjoining figure, $\angle C$ = right angle and $\angle B = 2 \angle A$.

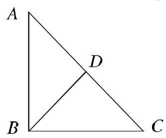
    Prove that, $AB = 2BC$.



15. Prove that, the exterior angle so formed by producing any side of a triangle is equal to the sum of the interior opposite angles.

16. Prove that, the difference between any two sides of a triangle is less than the third.

17. In the adjoining figure, $\angle B$ = right angle and D is the middle point of the hypotenuse AC of the triangle ABC. Prove that. $BD = \frac{1}{2} AC$.



18. In the ΔABC, $AB > AC$ and the bisector AD of $\angle A$ intersects BC at D. Prove that $\angle ADB$ is an obtuse angle.

19. Show that, any point on the perpe ndicular bisector of a line segment is equidistant from the terminal points of that line segment.

20. In the rightangled triangle $\angle A$ = right angle and D is the mid point of BC.

a. Draw a triangle ABC with given information.

b. Prove that $AB + AC > 2AD$

c. Prove that, $AD = \frac{1}{2} BC$

Chapter Seven

Practical Geometry

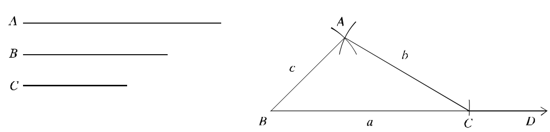
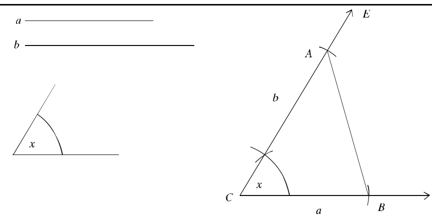
In the previous classes geometrical figures were drawn in proving different propositions and in the exercises. There was no need of precision in drawing these figures. But sometimes precision is necessary in geometrical constructions. For example, when an architect makes a design of a house or an engineer draws different parts of a machine, high precision of drawing is required. In such geometrical constructions, one makes use of ruler and compasses only. We have already learnt how to construct triangles and quadrilaterals with the help of ruler and compasses. In this chapter we will discuss the construction of some special triangles and quadrilaterals.

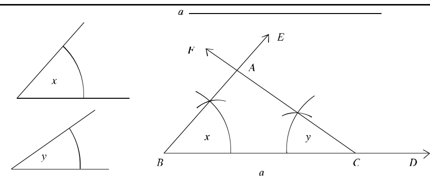
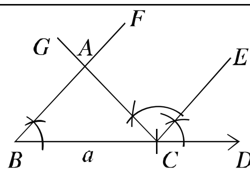
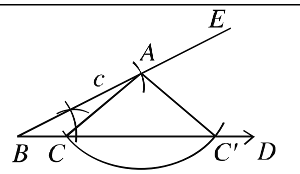
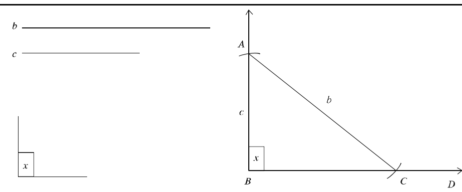
At the end of the chapter, the students will be able to –

- Explain triangles and quadrilaterals with the help of figures
- Construct triangle by using given data
- Construct parallelogram by using given data.

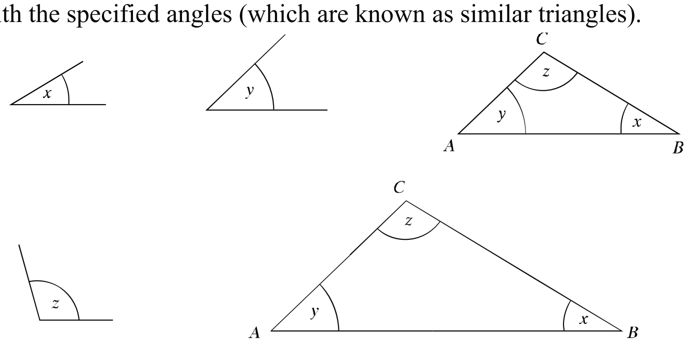
7·1 Construction of Triangles

Every triangle has three sides and three angles. But, to specify the shape and size of a triangle, all sides and angles need not to be specified. For example, as sum of the three angles of a triangle is two right angles, one can easily find the measurement of the third angle when the measurement of the two angles of the triangle given. Again, from the theorems on congruence of triangles it is found that the following combination of three sides and angles are enough to be congruent. That is, a combination of these three parts of a triangle is enough to construct a unique triangle. In class seven we have learnt how to construct triangles from the following data:

(1) Three sides	
(2) Two sides and their included angle.	

(3) Two angles and their adjacent sides	
(4) Two angles and an opposite side	
(5) Two sides and an opposite angle	
(6) Hypotenuse and a side of a rightangled triangle	

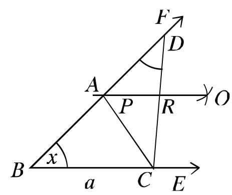
Observe that in each of the cases above, three parts of a triangle have been specified. But any three parts do not necessarily specify a unique triangle. As for example, if three angles are specified, infinite numbers of triangles of different sizes can be drawn with the specified angles (which are known as similar triangles).



Sometimes for construction of a triangle three such data are provided by which we can specify the triangle through various drawing. Construction in a few such cases is stated below.

Construction 1

The base of the base adjacent angle and the sum of other two sider of a triangle are given. Construct the triangle.



Let the base a, a base adjacent angle $\angle x$ and the sum s of the other two sides of a triangle ABC be given. It is required to construct it.

Steps of construction :

(1) From any ray BE cut the line segment BC equal to a. At B of the line segment BC, draw an angle $\angle CBF = \angle x$.

(2) Cut a line segment BD equal to s from the ray BF.

(3) Join C,D and at C make an angle $\angle DCG$ equal to $\angle BDC$ on the side of DC in which B lies.

(4) Let the ray CG intersect BD at A.

Then, ABC is the required triangle.

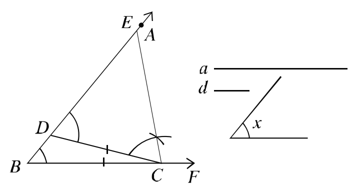
Proof : In ΔACD, $\angle ADC = \angle ACD$ [by construction]

$\therefore \ AC = AD.$

Now, In ΔABC, $\angle ABC = \angle x$, $BC = a$, [by construction]

and $BA + AC = BA + AD = BD = s$. Therefore, ΔABC is the required triangle.

Alternate Method



Let the base a, a base adjacent angle $\angle x$ and the sum s of the other two sides of a triangle ABC be given. It is required to construct the triangle.

Steps of construction:

(1) From any ray BE cut the line segment BC equal to a. At B of the line segment BC draw an angle $\angle CBF = \angle x$.

(2) Cut a line segment BD equal to s from the ray BF.

(3) Join C,D and construct the perpendicular bisector PQ of CD.

(4) Let the ray PQ intersect BD at A. Join A, C.

Then, ABC is the required triangle.

Proof: In ΔACR and ΔADR, $CR = DR$ $AR = AR$ and the included angle

$\angle ARC = \angle ARD$ [right angle]

$\Delta ACR \cong \Delta ADR$. $\therefore$ $AC = AD$

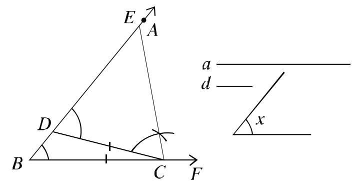
Now, In ΔABC, $\angle ABC = \angle x$, $BC = a$, [by construction]

and $BA + AC = BA + AD = BD = s$. Therefore, ΔABC is the required triangle.

Construction 2

The base of a triangle the base adjacent an acute angle and the difference of the other two sides are given. Construct the triangle.

Let the base a, a base adjacent angle $\angle x$ and the difference d of the other two sides of a triangle ABC be given. It is required to construct the triangle.



Steps of Construction :

(1) From any ray BE, cut the line segment BC , equal to a. At B of the line segment BC draw an angle $\angle CBF = \angle x$.

(2) Cut a line segment BD equal to s from the ray BE.

(3) Join C,D and at C, make an angle $\angle DCA$ equal to $\angle EDC$ on the side of DC in which C lies. Let the ray CA intersect BE at A.

Then ABC is the required triangle.

Proof :, In ΔACD, $\angle ADC = \angle ACD$ [by construction]

$\therefore$ $AC = AD$.

So, the difference of two sides $AB - AC = AB - AD = BD = d$.

Now, In ΔABC, $BC = a, AB - AC = d$ and $\angle ABC = \angle x$. Therefore, ΔABC is the required triangle.

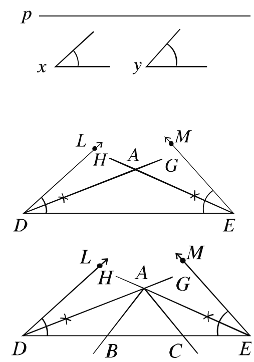
Activity :

1. If the given angle is not acute, the above construction is not possible. Why ? Explore any way for the construction of the triangle under such circumstances.
2. The base, the base adjacent angle and the difference of the other two sides of a triangle are given. Construct the triangle in an alternate method.

Construction 3

The base adjacent two angles and the perimeter of a triangle are given. Construct the triangle.

Let the base adjacent angles $\angle x$ and $\angle y$ and the perimeter p be geven. It is required to construct the triangle.



Steps of Construction :

(1) From any ray DF, cut the part DE equal to the perimeter p. Make angles $\angle EDL$ equal to $\angle x$ and $\angle DEM$ equal to $\angle y$ on the same side of the line segment DE at D and E.

(2) Draw the bisectors BG and EH of the two angles.

(3) Let these bisectors DG and EH intersect at a point A. At the point A, draw $\angle DAB$ equal to $\angle ADE$ and $\angle EAC$ equal to $\angle AED$.

(4) Let AB intersect DE at B and AC intersect DE at C.

Then, ΔABC is the required triangle.

Proof : In ΔADB, $\angle ADB = \angle DAB$ [by construction] $\therefore\ AB = DB$.

Again, in ΔACE, $\angle AEC = \angle EAC$; $\therefore\ CA = CE$.

Therefore, in ΔABC, $AB + BC + CA = DB + BC + CE = DE = p$.

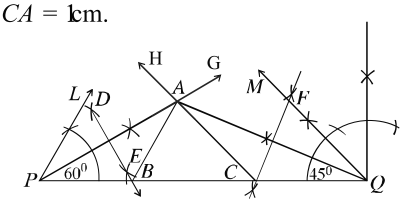
$$\angle ABC = \angle ADB + \angle DAB = \frac{1}{2}\angle x + \frac{1}{2}\angle x = \angle x$$

and $\angle ACB = \angle AEC + \angle EAC = \frac{1}{2}\angle y + \frac{1}{2}\angle y = \angle y$. Therefore, ΔABC is the required triangle.

Activity:

1. Two acute base adjacent angles and the perimeter of a triangle are given. Construct the triangle in an alternative way.

Example 1. Construct a triangle ABC, in which $\angle B = 60^\circ$, $\angle C = 45^\circ$ and the perimeter $AB + BC + CA = 1$ cm.



Steps of Construction: Follow the steps below :

(1) Draw a line segment $PQ = 1$ cm.

(2) At P, construct an angle of $\angle QPL = 60^\circ$ and at Q, an angle of $\angle PQM = 45^\circ$ on the same side of PQ.

(3) Draw the bisectors PG and QH of the two angles. Let the bisectors PG and QH of these angles intersect at A.

(4) Draw perpendicular bisector of the segments PA of QA to intersect PQ at B and C.

(5) Join A, B and A, C.

Then, ABC is the required triangle.

Activity : An adjacent side with the right angle and the difference of hypotenuse and the other side of a right-angled triangle are given. Construct the triangle.

Exercise 7.1

1. **Construct a triangle with the following data:**
   (a) The lengths of three sides are 3 cm, 3·5 cm, 2·8 cm.
   (b) The lengths of two sides are 4 cm , 3 cm and the included angle is 60°.
   (c) Two angles are 60° and 45° and their included side is 5 cm.
   (d) Two angles are 60° and 45° and the side opposite the angle 45° is 5 cm.
   (e) The lengths of two sides are 4.5 cm and 3.5 respectively and the angle opposite to the second side is 4 cm.
   (f) The lengths of the hypotenuse and a side are 6 cm and 4 cm respectively.
2. **Construct a triangle *ABC* with the following data:**
   (a) Base 3.5 cm, base adjacent angle 60° and the sum of the two other sides 8 cm.
   (b) Base 4 cm, base adjacent angle 50° and the sum of the two other sides 7.5 cm.
   (c) Base 4 cm, base adjacent angle 50° and the difference of the two other sides 1.5 cm.
   (d) Base 5 cm, base adjacent angle 45° and the difference of the two other sides 1 cm.
   (e) Base adjacent angles 60° and 45° and the perimeter 12 cm.
   (f) Base adjacent angles 30° and 45° and the perimeter 10 cm.
3. Construct a triangle when the two base adjacent angles and the length of the perpendicular from the vertex to the base are given.
4. Construct a right-angled triangle when the hypotenuse and the sum of the other two sides are given.
5. Construct a triangle when a base adjacent angle, the altitude and the sum of the other two sides are given.
6. Construct an equilateral triangle whose perimeter is given.
7 The base, an obtuse base adjacent angle and the difference of the other two sides of a trangle are given. Construcet the triangle.

7·2 Construction of Quadrilaterals

We have seen if three independent data are given, in many cases it is possible to construct a definite triangle. But with four given sides the construction of a definite quadrilateral is not possible. Five independent data are required for construction of a definite quadrilateral. A definite quadrilateral can be constructed if any one of the following combinations of data is known :

(a) Four sides and an angle

(b) Four sides and a diagonal

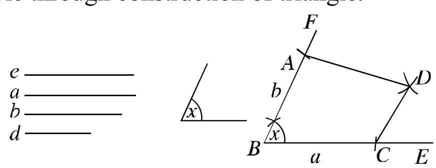
(c) Three sides and two diagonals

(d) Three sides and two included angles

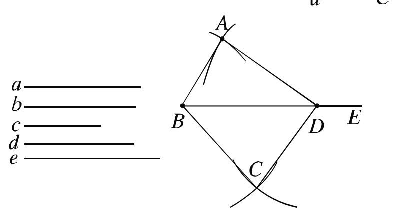
(e) Two sides and three angles.

In class VII, the construction of quadrilaterals with the above specified data has been discussed. If we closely look at the steps of construction, we see that in some cases it is possible to construct the quadrilaterals directly. In some cases, the construction is done by constructions of triangles. Since a diagonal divides the quadrilateral into two triangles, when one or two diagonals are included in data, construction of quadrilaterals is possible through construction of triangle.

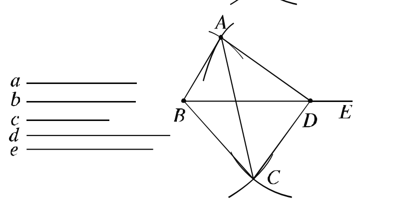
(1) Four sides and an angle



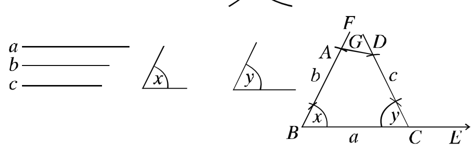
(2) Four sides and a diagonal



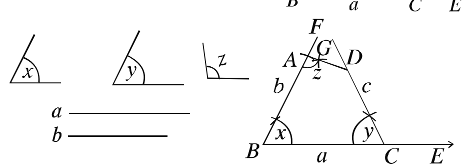
(3) Three sides and two diagonals



(4) Three sides and two included angles



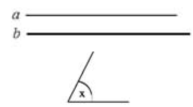
(5) Two sides and three angles



Sometimes special quadrilaterals can be constructed with fewer data. In such a cases, from the properties of quadrilaterals, we can retrieve five necessary data. For example, a parallelogram can be constructed if only the two adjacent sides and the included angle are given. In this case, only three data are given. Again, a square can be constructed when only one side of the square is given. The four sides of a square are equal and an angle is a right angle; so five data are easily specified.

Construction 4

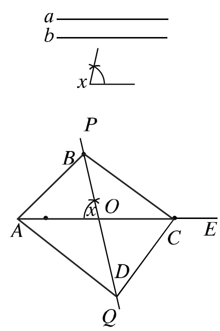
Two diagonals and an included angle between them of a parallelogram are given. Construct the parallelogram.



Let a and b be the diagonals of a parallelogram and $\angle x$ be an angle included between them. The parallelogram is to be constructed.

Steps of construction:

From any ray AE, cut the line segment $AC = a$. Bisect the line segment AC to find the mid-point O, At O construct the angle $\angle AOP = \angle x$ and extend the ray PO to the opposite ray OQ. From the rays OP and OQ cut two line segments OB and OD equal to $\frac{1}{2}$ b. Join A,B; A,D; C,B and C,D. Then $ABCD$ is the required parallelogram.



Proof: In triangles ΔAOB and ΔCOD,

$OA = OC = \frac{1}{2}a,\ OB = OD = \frac{1}{2}b$ [by construction]

and included $\angle AOB$ = included $\angle COD$ [opposite angle]

Therefore, $\Delta AOB \cong \Delta COD$.

So, $AB = CD$

and $\angle ABO = \angle CDO$;but the two angles are alternate angles.

$\therefore$ AB and CD are parallel and equal.

Similarly, AD and BC are parallel and equal.

Therefore, $ABCD$ is a parallelogram with diagonals

$$AC = AO + OC = \frac{1}{2}a + \frac{1}{2}a = a$$

and $BD = BO + OD = \frac{1}{2}b + \frac{1}{2}b = b$ and the angle included between the diagonals $\angle AOB = \angle x$.

Therefore, $ABCD$ is the required parallelogram.

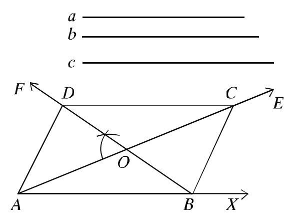
Construction 5

Two diagonals and a side of a parallelogram are given. Construct the parallelogram.

Let a and b be the diagonals and c be a side of the parallelogram. The parallelogram is to be constructed.

Steps of construction:

Bisect the diagonals a and b to locate their midpoints. From any ray AX, cut the line segment $AB = a$. With centre at A and B draw two arcs with radii $\frac{2}{a}$ and $\frac{b}{2}$ respectively on the same side of AB. Let the arcs intersect at O. Join A, O and O, B. Extend AO and BO to AE and BF respectively. Now cut $OC = \frac{2}{a}$ and $OD = \frac{b}{2}$ from OE and OF respectively. Join $A,D; D,C; C,B$. Then $ABCD$ is the required parallelogram.



Proof: In ΔAOB and ΔCOD,

$OA = OC = \frac{a}{2}; OB = OD = \frac{b}{2}$, [by construction]

and included $\angle AOB$ = included $\angle COD$ [opposite angle]

$\therefore \quad \Delta AOB \cong \Delta COD$.

$\therefore \quad AB = CD$ and $\angle ABO = \angle ODC$; but the angles are alternate angles.

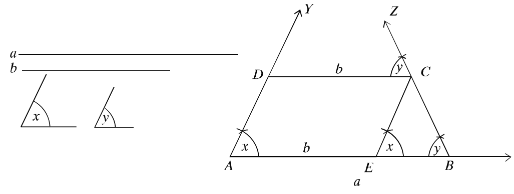
$\therefore \quad AB$ and CD are parallel and equal.

Similarly, AD and BC are parallel and equal.

Therefore, $ABCD$ is the required parallelogram.

Example 1. The parallel sides and two angles included with the larger side of a trapezium are given. Construct the trapezium.

Let a and b be the parallel sides of a trapezium where $a > b$ and $\angle x$ and $\angle y$ be two angles included with the side a. The trapezium is to be constructed.



Steps of construction:

From any ray AX, cut the line segment $AB = a$. At A of the line segment AB, construct the angle $\angle BAY = \angle x$ and at B, construct the angle $\angle ABZ = \angle y$. From the line segment AB, cut a line segment $AE=b$. Now at E, construct $BC \parallel AY$ which cuts BZ at C. Now construct $CD \parallel BA$. The line segment CD intersects the ray AY at D. Then $ABCD$ is the required trapezium.

Proof : By construction, $AB \parallel CD$ and $AD \parallel EC$. Therefore, $AECD$ is a parallelogram and $CD = AE = b$. Now in the quadrilateral $ABCD$, $AB=a$, $CD=b$, $AB \parallel CD$ and $\angle BAD = \angle x$, $\angle ABC = \angle y$ (by construction). Therefore, $ABCD$ is the required trapezium.

Activity : The perimeter and an angle of a rhombus are given. Construct the rhombus.

Exercise 7·2

1. The two angles of a right angled triangle are given. Which one of the following combination allows constructing the triangle?

   a. 63° and 36° b. 30° and 70° c. 40° and 50° d. 80° and 20°

2. i. A rectangle is a parallelogram

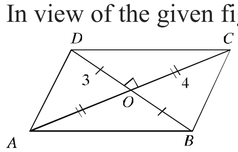
   ii. A square is a rectangle

   iii. A rhombus is a square

   On the basis of the above information, which one of the following is true?

   a. i and ii b. i and iii c. ii and iii d. i, ii and iii

   In view of the given figure, answer the questions 3 and 4.



3. What is the area of ΔAOB?

   a. 6 sq. units b. 7 sq. units c. 12 sq. units d. 14 sq. units

4. The perimeter of the quadrilateral is

   a. 12 units b. 14 units c. 20 units d. 28 units

5. **Construct a quadrilateral with the following data :**

   (a) The lengths of four sides are 3 cm, 3·5 cm, 2·5 cm, 3cm and an angle is 45°.

   (b) The lengths of four sides are 3.5 cm, 4 cm, 2·5 cm 3.5 cm and a diagonal is 5 cm.

   (c) The lengths of three sides are 3·2 cm, 3 cm, 3·5 cm and two diagonals are 2·8 cm, and 4.5 cm.

6. Construct a parallelogram with the following data:
   (a) The lengths of two diagonals are 4 cm, 6·5 cm and the included angle is $45°$.
   (b) The lengths of two diagonals are 5 cm, 6·5 cm and the included angle is $30°$.
   (c) The length of a side is 4 cm and the lengths of two diagonals are 5 cm and 6·5 cm.
   (d) The length of a side is 5 cm and the lengths of two diagonals are 4.5 cm and 6 cm.
7. The sides AB and BC and the angles $\angle B$, $\angle C$, $\angle D$ of the quadrilateral $ABCD$ are given. Construct the quadrilateral.
8. The four segments made by the intersecting points of the diagonals of a parallelogram and an included angle between them are $OA = 4$ cm. $OB = 5$ cm. $OC = 3·5$ cm $OD = 4·5$ cm and $\angle AOB = 80°$ respectively. Construct the quadrilateral.
9. The length of a side of a rhombus and an angle are 3.5 cm. and $45°$ respectively; construct the rhombus.
10. The length of a side and a diagonal of a rhombus are given; construct the rhombus.
11. The length of two diagonals of a rhombus are given. Construct the rhombus.
12. The perimeter of a square is given. Construct the square.
13. The houses of Mr. Zaki and Mr. Zafrul are in the same boundary and the area of their house is equal. But the house of Zoki is rectangular and the house of Mr. Zafrul is in shape of parallelogram.
    (a) Construct the boundary of each of their houses taking the length of base 10 units and height 8 units.
    (b) Show that the area of the house of Mr. Zaki is less than the area of the house of Mr. Zafrul.
    (c) If the ratio of the length and the breadth of the house of Mr. Zaki is 4:3 and its area is 300 sq. units, find the ratio of the area of their houses.
14. The lengths of the hypotenuse and a side of right angled triangle are 7 cm and 4 cm. Use the information to answer the following questions :
    a. Find the length of the other side of the triangle.
    b. Construct the triangle.
    c. Construct a square whose perimeter is equal to the perimeter of the triangle.
15. $AB = 4$ cm, $BC = 5$ cm $\angle A = 85°$, $\angle B = 80°$ and $\angle C = 95°$. of the quadrilateral $ABCD$. Use the information to answer the following questions:
    a. Construct a rhombus and give the name.
    b. Use the above information to construct the quadrilateral $ABCD$.
    c. Construct an equilateral triangle whose perimeter is equal to the perimeter of the quadrilateral $ABCD$.

Chapter Eight

Circle

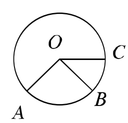
We have already known that a circle is a geometrical figure in a plane consisting of points equidistant from a fixed point. Different concepts related to circles like centre, diameter, radius, chord etc has been discussed in previous class. In this chapter, the propositions related to arcs and tangents of a circle in the plane will be discussed.

At the end of the chapter, the students will be able to

- Explain arcs, angle at the centre, angle in the circle, quadrilaterals inscribed in the cirlce
- Prove theorems related to circle
- State constructions related to circle.

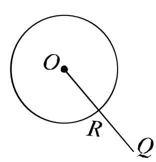
8·1 Circle

A circle is a geometrical figure in a plane whose points are equidistant from a fixed point. The fixed point is the centre of the circle. The closed path traced by a point that keeps it distance from the fixed centre is a circle. The distance from the centre is the radius of the circle.



Let O be a fixed point in a plane and r be a fixed measurement. The set of points which are at a distance r from O is the circle with centre O and radius r. In the figure, O is the centre of the circle and A, B and C are three points on the circle. Each of OA, OB and OC is a radius of the circle. Some coplanar points are called concylcic if a circle passes through these points, i.e. there is a circle on which all these points lie. In the above figure, the points A, B and C are concyclic.

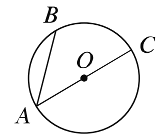
Interior and Exterior of a Circle



If O is the centre of a circle and r is its radius, the set of all points on the plane whose distances from O are less than r, is called the interior region of the circle and the set of all points on the plane whose distances from O are greater than r, is called the exterior region of the circle. The line segment joining two points of a circle lies inside the circle.

The line segment drawn from an interior point to an exterior point of a circle intersects a circle at one and only one point. In the figure, P and Q are interior and exterior points of the circle respectively. The line segment PQ intersects the circle at R only.

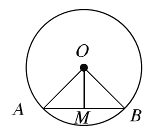
Chord and Diameter of a Circle



The line segment connecting two different points of a circle is a chord of the circle. If the chord passes through the centre it is known as diameter. That is, any chord forwarding to the centre of the circle is diameter. In the figure, *AB* and *AC* are two chords and *O* is the centre of the circle. The chord *AC* is a diameter, since it passes through the centre. *OA* and *OC* are two radii of the circle. Therefore, the centre of a circle is the mid-point of any diameter. The length of a diameter is 2*r*, where *r* is the radius of the circle.

Theorem 1

The line segment drawn from the centre of a circle to bisect a chord other than diameter is perpendicular to the chord.



Let *AB* be a chord (other than diameter) of a circle *ABC* with centre *O* and *M* be the midpoint of the chord. Join *O, M*. It is to be proved that the line segment *OM* is perpendicular to the chord *AB*.

Construction: Join *O, A* and *O, B*.

Proof:

Steps	Justification
(1) In $\Delta\angle OAM$ and ΔOBM,	
$OA = OB$	[*M* is the mid point of *AB*]
$AM = BM$	[radius of same circle]
and $OM = OM$	[common side]
Therefore, $\Delta OAM \cong \Delta OBM$	[SSS theorem]
$\therefore\ \angle OMA = \angle OMB$	
(2) Since the two angles are equal and together maka a straight angle.	
$\angle OMA = \angle OMB = 1$ right angle.	
Therefore, $OM \perp AB$. (Proved).	

Corollary 1: The perpendicular bisector of any chord passes through the centre of the circle.

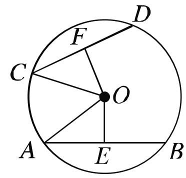
Corollary 2: A straight line can not intersect a circle in more than two points.

Activity :

1. The theorem opposite of the theorem 1 states that the perpendicular from the centre of a circle to a chord bisects the chord. Prove the theorem.

Theorem 2

All equal chords of a circle are equidistant from the centre.



Let *AB* and *CD* be two equal chords of a circle with centre *O*. It is to be proved that the chords *AB* and *CD* are equidistant from the centre.

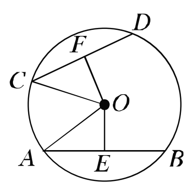
Construction: Draw from *O* the perpendiculars *OE* and *OF* to the chords *AB* and *CD* respectively. Join *O, A* and *O, C.*

Proof:

Steps	Justification
(1) $OE \perp AB$ and $OF \perp CD$ Therefore, $AE = BE$ and $CF = BF$. $\therefore AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$	[The perpendicular from the centre bisects the chord]
(2) But $AB = DC$ $\therefore AE = CF$	[supposition]
(3) Now in the right-angled triangles ΔOAE and ΔOCF hypotenuse OA = hypotenuse OC and $AE = CF$ $\therefore \Delta OAE \cong \Delta OCF$ $\therefore OE = OF$	 [radius of same circle] [Step 2] [RHS theorem]
(4) But *OE* and *OF* are the distances from *O* to the chords *AB* and *CD* respectively. Therefore, the chords *AB* and *CD* are equidistant from the centre of the circle. (Proved)	

Theorem 3

Chords equidistant from the centre of a circle are equal.



Let *AB* and *CD* be two chords of a circle with centre *O*. *OE* and *OF* are the perpendiculars from *O* to the chords *AB* and *CD* respectively. Then *OE* and *OF* represent the distance from centre to the chords *AB* and *CD* respectively.

It is to be proved that if $OE = OF$, $AB = CD$.

Construction: Join *O, A* and *O, C.*

Proof:

Steps	**Justification**
(1) Since $OE \perp AB$ and $OF \perp CD$. Therefore, $\angle OEA = \angle OFC =$ 1 right angle	[right angles]
(2) Now in the right angled triangles ΔOAE and ΔOCF	
hypotenuse OA = hypotenuse OC and $OE = OF$	[radius of same circle]
$\therefore \quad \Delta OAE \cong \Delta OCF$	[RHS theorem]
$\therefore \quad AE = CF$.	
(3) $AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$	[The perpendicular from the centre bisects the chord]
(4) Therefore $\frac{1}{2} AB = \frac{1}{2} CD$	
i.e., $AB = CD$ (Proved)	

Corollary 1: The diameter is the greatest chord of a circle.

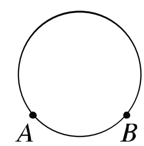
Exercise 8.1

1. Prove that if two chords of a circle bisect each other, their point of intersection is the centre of the circle.
2. Prove that the straight line joining the middle points of two parallel chords of a circle pass through the centre and is perpendicular to the chords.
3. Two chords AB and AC of a circle subtend equal angles with the radius passing through A. Prove that, $AB = AC$.
4. In the figure, O is the centre of the circle and chord AB = chord AC. Prove that $\angle BAO = \angle CAO$.
5. A circle passes through the vertices of a right angled triangle. Show that, the centre of the circle is the middle point of the hypotenuse.
6. A chord AB of one of the two concentric circles intersects the other circle at points C and D. Prove that, $AC = BD$.
7. If two equal chords of a circle intersect each other, show that two segments of one are equal to two segments of the other.
8. Prove that, the middle points of equal chords of a circle are concyclic.
9. Show that, the two equal chords drawn from two ends of the diameter on its opposite sides are parallel.

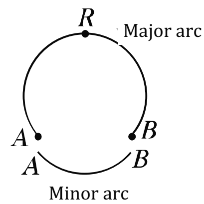
10. Show that, the two parallel chords of a circle drawn from two ends of a diameter on its opposite sides are equal.

11. Show that, of the two chords of a circle the bigger chord is nearer to the centre than the smaller.

8.2 The arc of a circle

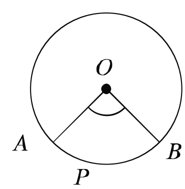


An arc is the piece of the circle between any two points of the circle. Look at the pieces of the circle between two points *A* and *B* in the figure. We find that there are two pieces, one comparatively large and the other small. The large one is called the *major arc* and the small one is called the *minor arc.*



A and *B* are the terminal points of this arc and all other points are its internal points. With an internal fixed point *C* the arc is called arc *ABC* and is expressed by the symbol *ACB*. Again, sometimes minor arc is expressed by the symbol *AB*. The two points *A* and *B* of the circle divide the circle into two arcs. The terminal points of both arcs are *A* and *B* and there is no other common point of the two arcs other than the terminal points.

Arc cut by an Angle



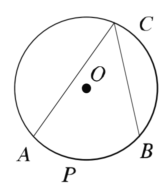
An angle is said to cut an arc of a circle if

(i) each terminal point of the arc lies on the sides of the angle

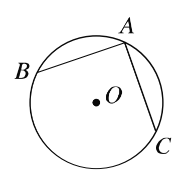
(ii) each side of the angle contains at least one terminal point

(iii) Every interior point of the arc lies inside the angle. The angle shown in the figure cuts the *APB* arc of the circle with centre O.

Angle in a Circle



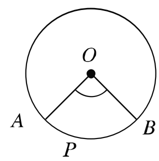
If the vertex of an angle is a point of a circle and each side of the angle contains a point of the circle, the angle is said to be an angle in the circle or an angle inscribed in the circle. The angles in the figure are all angles in a circle. Every angle in a circle cuts an arc of the circle. This arc may be a major or minor arc or a semicircle.



The angle in a circle cuts an arc of the circle and the angle is said to be standing on the cut off arc. The angle is also known as the angle inscribed in the conjugate arc. In the adjacent figure, the angle stands on the arc *APB* and is inscribed in the conjugate arc *ACB*. It is to be noted that *APB* and *ACB* are mutually conjugate.

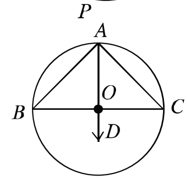
Remark: The angle inscribed in an arc of a circle is the angle with vertex in the arc and the sides passing through the terminal points of the arc. An angle standing on an arc is the angle inscribed in the conjugate arc.

Angle at the Centre



The angle with vertex at the centre of the circle is called an angle at the centre. An angle at the centre cuts an arc of the circle and is said to stand on the arc. In the adjacent figure, $\angle AOB$ is an angle at the centre and it stands on the arc APB.

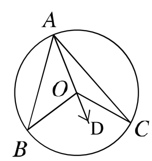
Every angle at the centre stands on a minor arc of the circle. In the figure APB is the minor arc. So the vertex of an angle at the centre always lies at the centre and the sides pass through the two terminal points of the arc.



To consider an angle at the centre standing on a semi-circle the above description is not meaningful. In the case of semi-circle, the angle at the centre $\angle BOC$ is a straight angle and the angle on the arc $\angle BAC$ is a right angle.

Theorem 4

The angle subtended by the same arc at the centre is double of the angle subtended by it at any point on the remaining part of the circle.



Given an arc BC of a circle subtending angles $\angle BOC$ at the centre O and $\angle BAC$ at a point A of the circle ABC. We need to prove that $\angle BOC = 2\ \angle BAC$.

Construction: Suppose, the line segment AC does not pass through the centre O. In this case, draw a line segment AD at *A passing through the centre O*.

Proof :

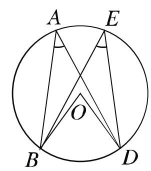
Steps	Justification
(1) In ΔAOB, the external angle $\angle BOD = \angle BAO + \angle ABO$	[An exterior angle of a triangle is equal to the sum of the two interior opposite angles.]
(2) Also in $\Delta\ AOB$, $OA = OB$	[Radius of a circle]
Therefore, $\angle BAO = \angle ABO$	[Base angles of an isosceles triangle are equal]
(3) From steps (1) and (2), $\angle BOD = 2\angle BAO$.	
(4) Similarly, $\angle\ COD = 2\ \angle\ CAO$	
(5) From steps (3) and (4), $\angle BOD + \angle COD = 2\angle BAO + 2\angle CAO$	[by adding]
This is the same as $\angle BOC = 2\angle BAC$. [Proved]	

We can state the theorem in a different way. The angle standing on an arc of the circle is half the angle subtended by the arc at the centre.

Activity : Prove the theorem 4 when *AC* passes through the centre of the circle *ABC*.

Theorem 5

Angles in a circle standing on the same arc are equal.



Let *O* be the centre of a circle and standing on the arc *BD*, $\angle BAD$ and $\angle BED$ be the two angles in the circle. We need to prove that $\angle BAD = \angle BED$.

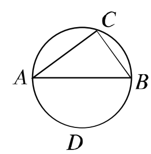
Construction : Join *O, B* and *O, D*.

Proof :

Steps	Justification
(1) The arc *BD* subtends an angle $\angle BOD$ at the centre *O*. Therefore, $\angle BOD = 2\angle BAD$ and $\angle BOD = 2\angle BED$ $\therefore\ 2\angle BAD = 2\angle BED$ or, $\angle BAD = \angle BED$ (Proved)	[The angle subtended by an arc at the centre is double of the angle subtended on the circle]

Theorem 6

The angle in the semi- circle is a right angle.



Let *AB* be a diameter of circle with centre at *O* and $\angle ACB$ is the angle subtended by a semicircle. It is to be proved that $\angle ACB = 1$ right angle.

Construction: Take a point *D* on the circle on the opposite side of the circle where *C* is located.

Proof:

Steps	Justification
(1) The angle standing on the arc *ADB* $\angle ACB = \frac{1}{2}$ (straight angle in the centre $\angle AOB$) (2) But the straight angle $\angle AOB$ is equal to 2 right angles. $\angle ACB = \frac{1}{2}$ (2 right angles) = 1 right angle. (Proved)	[The angle standing on an arc at any point of the circle is half the angle at the centre]

Corollary 1. The circle drawn with hypotenuse of a right-angled triangle as diameter passes through the vertices of the triangle.

Corollary 2. The angle inscribed in the major arc of a circle is an acute angle.

Activity :

1 Prove that any angle inscribed in a minor arc is obtuse.

Exercise 8.2

1 *ABCD* is a quadrilateral inscribed in a circle with centre *O*. If the diagonals *AB* and *CD* intersect at the point *E*, prove that $\angle AOB + \angle COD = 2\,\angle AEB$.

2 Two chords *AB* and *CD* of the circle *ABCD* intersect at the point *E*. Show that, ΔAED and ΔBEC are equiangular.

3. In the circle with centre *O* $\angle ADB + \angle BDC =$ 1 right angle. Prove that, *A, B* and *C* lie in the same straight line.

4. Two chords *AB* and *CD* of a circle intersect at an interior point. Prove that, the sum of the angles subtended by the arcs *AC* and *BD* at the centre is twice $\angle AEC$.

5. Show that, the oblique sides of a cyclic trapezium are equal.

6. *AB* and *CD* are the two chords of a circle ; *P* and *Q* are the middle points of the two minor arcs made by them. The chord *PQ* intersects the chords *AB* and *AC* at points *D* and *E* respectively. Show that, $AD = AE$.

8·3 Quadrilateral inscribed in a circle

An inscribed quadrilateral or a quadrilateral inscribed in a circle is a quadrilateral having all four vertices on the circle. Such quadrilaterals possess a special property. The following avtivity helps us understand this property.

Activity:

Draw a few inscribed quadrilaterals ABCD. This can easily be accomplished by drawing circles with different radius and then by taking four arbitrary points on each of the circles. Measure the angles of the quadrilaterals and fill in the following table.

Serial No.	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A+\angle C$	$\angle B+\angle D$
1						
2						
3						
4						
5						

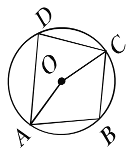
What to you infer from the table?

Circle related Theorems

Theorem 7

The sum of the two opposite angles of a quadrilateral inscribed in a circle is two right angles.

Let $ABCD$ be a quadrilateral inscribed in a circle with centre O. It is required to prove that, $\angle ABC + \angle ADC = 2$ right angles and $\angle BAD + \angle BCD = 2$ right angles.



Construction : Join O, A and O, C.

Proof :

Steps	Justification
(1) Standing on the same arc ADC, the angle at centre $\angle AOC = 2$ ($\angle ABC$ at the circumference) that is, $\angle AOC = 2 \angle ABC$.	[The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]
(2) Again, standing on the same arc ABC, reflex $\angle AOC$ at the centre = 2 ($\angle ADC$ at the circumference) that is, reflex $\angle AOC = 2 \angle ADC$ $\therefore \angle AOC +$ reflex $\angle AOC = 2(\angle ABC + \angle ADC)$ But $\angle AOC +$ reflex $\angle AOC = 4$ right angles $\therefore$ $2(\angle ABC + \angle ADC) = 4$ right angles $\therefore$ $\angle ABC + \angle ADC = 2$ right angles. In the same way, it can be proved that $\angle BAD + \angle BCD = 2$ right angles. (Proved)	[The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]

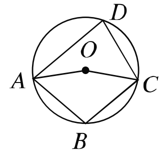
Corollary 1: If one side of a cyclic quadrilateral is extended, the exterior angle formed is equal to the opposite interior angle.

Corollary 2: A parallelogram inscribed in a circle is a rectangle.

Theorem 8

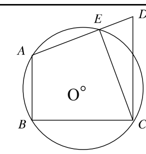
If two opposite angles of a quadrilateral are supplementary, the four vertices of the quadrilateral are concyclic.

Let $ABCD$ be the quadrilateral with $\angle ABC + \angle ADC = 2$ right angles, inscribed in a circle with centre O. It is required to prove that the four points A, B, C, D are concyclic.



Construction: Since the points A, B, C are not collinear, there exists a unique circle which passes through these three points. Let the circle intersect AD at E. Join A, E.

Proof :

Steps	Justification
(1) ABCE is a quadrilateral inscribed in the circle. Therefore, $\angle ABC + \angle AEC =$ 2 right angles. But $\angle ABC + \angle ADC =$ 2 right angles [given] $\therefore$ $\angle AEC = \angle ADC$ But this is impossible, since in ΔCED, exterior $\angle AEC >$ opposite interior $\angle ADC$ Therefore, *E* and *D* points can not be different points. So, *E* must coincide with the point *D*. Therefore, the points *A, B, C, D* are concyclic.	 [The sum of the two opposite angles of an inscribed quadrilateral is two right angles.] [The exterior angle is greater than any opposite interior angle.]

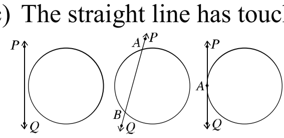
Exercise 8.3

1. If the internal and extern al bisectors of the angles $\angle B$ and $\angle C$ of ΔABC meet at P and Q respectively, prove that *B, P, C, Q* are concyclcic.
2. Prove that, the bisector of any angle of a cyclic quadrilateral and the exterior bisector of its opposite angle meet on the circumference of the circle.
3. *ABCD* is a circle. If the bisectors of $\angle CAB$ and $\angle CBA$ meet at the point *P* and the bisectors of $\angle DBA$ and $\angle DAB$ meet at *Q*, prove that, the four points *A, Q, P, B* are concyclic.
4. The chords *AB* and *CD* of a circle with centre *D* meet at right angles at some point within the circle, prove that, $\angle AOD + \angle BOC =$ 2 right angles.
5. If the vertical angles of two triangles standing on equal bases are supplementary, prove that their circumeircles are equal.
6. The opposite angles of the quadrilateral *ABCD* are supplementary to each other. If the line *AC* is the bisector of $\angle BAD$, prove that, *BC = CD.*

8·4 Secant and Tangent of the circle

Consider the relative position of a circle and a straight line in the plane. Three possible situations of the following given figures may arise in such a case:

(a) The circle and the straight line have no common points
(b) The straight line has cut the circle at two points
(c) The straight line has touched the circle at a point.

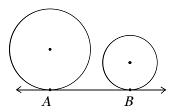


A circle and a straight line in a plane may at best have two points of intersection. If a circle and a straight line in a plane have two points of intersection, the straight line is called a secant to the circle and if the point of intersection is one and only one, the straight line is called a tangent to the circle. In the latter case, the common point is called the point of contact of the tangent. In the above figure, the relative position of a circle and a straight line is shown. In figure (i) the circle and the straight line PQ have no common point; in figure (ii) the line PQ is a secant, since it intersects the circle at two points A and B and in figure (iii) the line PQ has touched the circle at A. PQ is a tangent to the circle and A is the point of contact of the tangent.

Remark : All the points between two points of intersection of every secants of the circle lie interior of the circle.

Common tangent

If a straight line is a tangent to two circles, it is called a common tangent to the two circles. In the adjoining figures, AB is a common tangent to both the circles. In figure (a) and (b), the points of contact are different. In figure (c) and (d), the points of contact are the same.



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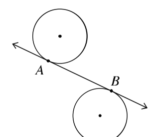
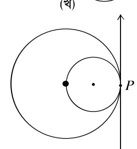


(খ)

If the two points of contact of the common tangent to two circles are different, the tangent is said to be

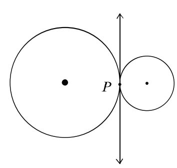
(a) direct common tangent if the two centres of the circles lie on the same side of the tangent and

(b) transverse common tangent, if the two centres lie on opposite sides of the tangent.



(গ)

The tangent in figure (a) is a direct common one and in figure (b) it is a transverse common tangent.



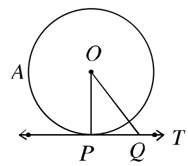
If a common tangent to a circle touches both the circles at the same point, the two circles are said to touch each other at that point. In such a case, the two circles are said to have touched internally or externally according to their centres lie on the same side or opposite side of the tangent. In figure (c) the two circles have touched each other internally and in figure (d) externally.

Theorem 9

The tangent drawn at any point of a circle is perpendicular to the radius through the point of contact of the tangent.

Let PT be a tangent at the point P to the circle with centre O and OP is the radius throug0h the point of contact. It is required to prove that, $PT \perp OP$.

Let PT be a tangent at the point P to the circle with centre O and OP is the radius throug0h the point of contact. It is required to prove that, $PT \perp OP$.



Construction: Take any point Q on PT and join O, Q.

Proof :

Since PT is a tangent to the circle at the point P, hence every point on it except P lies outside the circle. Therefore, the point Q is outside of the circle.

OQ is greater than OP that is, $OQ > OP$ and it is true for every point Q on the tangent PT except P. So, OP is the shortest distance from the centre O to PT. Therefore, $PT \perp OP$. (Proved)

Corollary 1. At any point on a circle, only one tangent can be drawn.

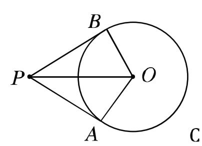
Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

Corollary 3. At any point of the circle the perpendicular to the radius is a tangent to the circle.

Theorem 10

If two tangents are drawn to a circle from an external point, the distances from that point to the points of contact are equal.

Let P be a point outside a circle ABC with centre O, and two line segments PA and PB be two tangents to the circle at points A and B. It is required to prove that. $PA = PB$.



Construction: Let us join O, A; O, B and O, P.

Proof:

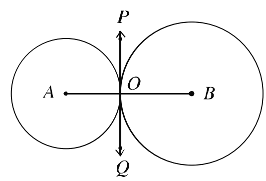
Steps	Justification
(1) Since PA is a tangent and OA is the radius through the point of tangent $PA \perp OA$. $\therefore$ $\angle PAO$ = 1 right angle Similarly, $\angle PBO$ = 1 right angle $\therefore$ both ΔPAO and ΔPBO are right-angled triangles. (2) Now in the right angled triangles ΔPAO and ΔPBO, hypotenuse PO = hypotenuse PO,	[The tangent is perpendicular to the radius through the point of contact of the tangent]

Remarks:

1. If two circles touch each other externally, all the points of one excepting the point of contact will lie outside the other circle.
2. If two circles touch each other internally, all the points of the smaller circle excepting the point of contact lie inside the greater circle.

Theorem 11

If two circles touch each other externally, the point of contact of the tangent and the centres are collinear.



Let the two circles with centres at A and B touch each other externally at O. It is required to prove that the points A, O and B are collinear.

Construction: Since the given circles touch each other at O, they have a common tangent at the point O. Now draw the common tangent POQ at O and join O, A and O, B.

Proof: In the circles OA is the radius through the point of contact of the tangent and POQ is the tangent.

Therefore $\angle POA = 1$ right angle. Similarly $\angle POB = 1$ right angle

Hence $\angle POA + \angle POB = 1$ right angle $+ 1$ right angle $= 2$ right angles

or $\angle AOB = 2$ right angles i.e. $\angle AOB$ is a straight angle. $\therefore$ A, O and B are collinear. (Proved)

Corollary 1. If two circles touch each other externally, the distance between their centres is equal to the sum of their radii

Corollary 2. If two circles touch each other internally, the distance between their centres is equal to the difference of their radii.

Activity:

1. Prove that, if two circles touch each other internally, the point of contact of the tangent and the centres are collinear.

Exercise 8·4

1. Two tangents are drawn from an external point P to the circle with centre O. Prove that OP is the perpendicular bisector of the chord through the touch points.
2. Given that tangents PA and PB touches the circle with centre O at A and B respectively. Prove that PO bisects $\angle APB$.

3. Prove that, if two circles are concentric and if a chord of the greater circle touches the smaller, the chord is bisected at the point of contact.
4. *AB* is a diameter of a circle and *BC* is a chord equal to its radius. If the tangents drawn at *A* and *C* meet each other at the point *D*, prove that *ACD* is an equilateral triangle.
5. Prove that a circumscribed quadrilateral of a circle having the angles subtended by opposite sides at the centre are supplementary.

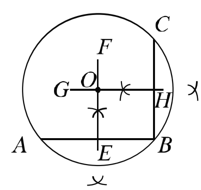
8·5 Constructions related to circles

Construction 1

To determine the centre of a circle or an arc of a circle.

Given a circle as in figure (a) or an arc of a circle as in figure (b). It is required to determine the centre of the circle or the arc.

Construction : In the given circle or the arc of the circle, three different points *A, B, C* are taken. The perpendicular bisectors *EF* and *GH* of the chords *AB* and *BC* are drawn respectively. Let the bisectors intersect at *O*. The *O* is the required centre of the circle or of the arc of the circle.



Proof: By construction, the line segments *EF* and *GH* are the perpendicular bisectors of chords *AB* and *BC* respectively. But both *EF* and *GH* pass through the centre and their common point is *O*. Therefore, the point *O* is the centre of the circle or of the arc of the circle.

Tangents to a Circle

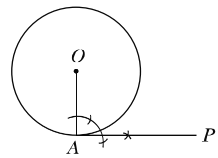
We have known that a tangent can not be drawn to a circle from a point internal to it. If the point is on the circle, a single tangent can be drawn at that point. The tangent is perpendicular to the radius drawn from the specified point. Therefore, in order to construct a tangent to a circle at a point on it, it is required to construct the radius from the point and then construct a perpendicular to it. Again, if the point is located outside the circle, two tangents to the circle can be constructed.

Construction 2

To draw a tangent at any point of a circle.

Let *A* be a point of a circle whose centre is *O*. It is required to draw a tangent to the circle at the point *A*.

Construction :



O, A are joined. At the point *A,* a perpendicular *AP* is drawn to *OA*. Then *AP* is the required tangent.

Proof: The line segment *OA* is the radius passing through *A* and *AP* is perpendicular to it. Hence, *AP* is the required tangent.

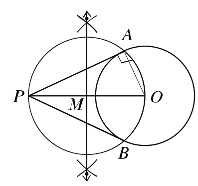
Remark : At any point of a circle only one tangent can be drawn.

Construction 3

To draw a tangent to a circle from a point outside.

Let *P* be a point outside of a circle whose centre is *O*. A tangent is to be drawn to the circle from the point *P*.

Construction :



(1) Join *P*, *O*. The middle point *M* of the line segment *PO* is determined.

(2) Now with *M* as centre and *MO* as radius, a circle is drawn. Let the new circle intersect the given circle at the points *A* and *B*.

(3) *A*, *P* and *B*, *P* are joined.

Then both *AP* or *BP* are the required tangents.

Proof: *A, O* and *B, O* are joined. *PO* is the diameter of the circle *APB*.

$\therefore$ $\angle PAO$ = 1 right angle [the angle in the semicircle is a right angle]

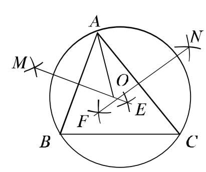
So the line segment *OA* is perpendicular to *AP*. Therefore, the line segment *AP* is a tangent at *A* to the circle with centre at *O*. Similarly the line segment *BP* is also a tangent to the circle.

Remark: Two and only two tangents can be drawn to a circle from an external point.

Construction 4

To draw a circle circumscribing a given triangle.

Let *ABC* be a triangle. It is required to draw a circle circumscribing it. That is, a circle which passes through the three vertices *A, B* and *C* of the triangle *ABC* is to be drawn.



Construction :

(1) *EM* and *FN* the perpendicular bisectors of AB and *AC* respectively are drawn. Let the line segments intersect each other at *O*.

(2) A, O are joined. With O as centre and radius equal to OA, a circle is drawn. Then the circle will pass through the points A, B and C and this circle is the required circumcircle of ΔABC.

Proof : B, O and C, O are joined. The point O stands on EM, the perpendicular bisector of AB.

$\therefore$ $OA = OB$. Similarly, $OA = OC$

$\therefore$ $OA = OB = OC$.

Hence, the circle drawn with O as the centre and OA as the radius passes through the three points A, B and C. This circle is the required circumcircle of ΔABC.

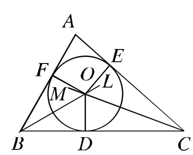
Activity:
In the above figure, the circumcircle of an acute angled triangle is constructed. Construct the circumcircle of an obtuse and right-angled triangles.

Notice that for in obtuse-angled triangle, the circumcentre lies outside the triangle, in acute-angle triangle, the circumcentre lies within the triangle and in right-angled triangle, the circumcentre lies on the hypotenuse of the triangle.

Construction 5

To draw a circle inscribed in a triangle.

Let ΔABC be a triangle. To inscribe a circle in it or to draw a circle in it such that it touches each of the three sides BC, CA and AB of the triangle.



Construction : BL and CM, the bisectors respectively of the angles $\angle ABC$ and $\angle ACB$ are drawn. Let the line segments intersect at O. OD is drawn perpendicular to BC from O and let it intersect BC at D. With O as centre and OD as radius, a circle is drawn. Then, this circle is the required inscribed circle.

Proof : From O, OE and OF are drawn perpendiculars respectively to AC and AB. Let these two perpendiculars intersect the respective sides at E and F. The point O lies on the bisector of $\angle ABC$.

$\therefore$ $OF = OD$.

Similarly, as O lies on bisector of $\angle ACB$, $OE = OD$

$\therefore$ $OD = OE = OF$

Hence, the circle drawn with centre as *O* and *OD* as radius passes through *D, E* and *F*.

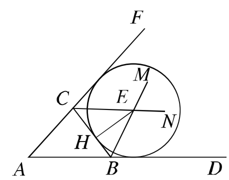
Again, *BC, AC* and *AB* respectively are perpendiculars to *OD, OE* and *OF* at their extremities. Hence, the circle lying inside ΔABC touches its sides at the points *D, E* and *F*.

Hence, the circle *DEF* is the required inscribed circle of ΔABC.

Construction 6

To draw an ex-circle of a given triangle.

Let *ABC* be a triangle. It is required to draw its excircle. That is, to draw a circle which touches one side of *ABC* and the other two sides produced.



Construction: Let *AB* and *AC* be produced to *D* and *F* respectively. *BM* and *CN,* the bisectors of $\angle DBC$ and $\angle FCB$ respectively are drawn. Let *E* be their point of intersection. From *E*, perpendicular *EH* is drawn on *BC* and let *EH* intersect *BC* at *H*. With *E* as centre and radius equal to *EH*, a circle is drawn.

The circle *HGL* is the excircle of the triangle *ABC*.

Proof : From *E*, perpendiculars *EG* and *EL* respectively are drawn to line segments *BD* and *CF*. Let the perpendicular intersect line segments *G* and *L* respectively. Since *E* lies on the bisector of $\angle DBC$

$$\therefore EH = EG$$

Similarly, the point E lies on the bisector of $\angle FCB$, so $EH = EL$

$$\therefore EH = EG = EL$$

Hence, the circle drawn with *E* as centre and radius equal to *EL* passes through *H, G* and *L*.

Again, the line segments *BC*, *BD* and *CF* respectively are perpendiculars at the extremities of *EH, EG* and *EL*. Hence, the circle touches the three line segments at the three points *H*, *G* and *L* respectively. Therefore, the circle *HGL* is the excircle of ΔABC.

Remark : Three excircles can be drawn with any triangle.

Activity: Construct the two other excircles of a triangle.

Exercise 8·5

1. Observe the following information:

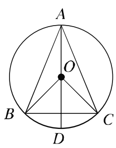
   i. The tangent to a circle is perpendicular to the radius to the point of contact.

   ii. The angle subtended in a semicircle is a right angle.

   iii. All equal chords of a circle are equidistant from the centre.

Which one of the following is correct ?

   (a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii



Use the above figure to answer questions 2 and 3:

2. $\angle BOD$ equals to

   a. $\frac{1}{2}\angle BAC$ b. $\frac{1}{2}\angle BAD$ c. $2\angle BAC$ d. $2\angle BAD$

3. The circle is of the triangle ABC

   a. inscribed circle b. circumscribed circle c. excircle d. ellipse

4. The angle inscribed in a major arc is

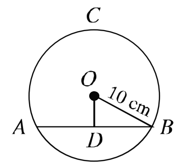
   a. acute angle b. right angle c. obtuse angle d. complementary angle

5. Draw a tangent to a circle which is parallel to a given straight line.
6. Draw a tangent to a circle which is perpendicular to a given straight line.
7. Draw two tangents to a circle such that the angle between them is 60°.
8. Draw the circumcircle of the triangle whose sides are 3 cm, 4 cm and 4.5 cm and find the radius of this circle.
9. Draw an excircle to an equilateral triangle ABC touching the side AC of the triangle, the length of each side being 5 cm.
10. Draw the inscribed and the circumscribed circles of a square.
11. Prove that two circles drawn on equal sides of an isosceles triangle as diameters mutually intersect at mid point of its base.
12. Prove that in a right-angled triangle, the length of line segment joining mid point of the hypotenuse to opposite vertex is half the hypotenuse.
13. ABC is a triangle. If the circle drawn with AB as diameter intersects BC at D, prove that the circle drawn with AC as diameter also passes through D.

15. If the chords AB and CD of a circles with centre O intersect at an internal point E, prove that $\angle AEC = \frac{1}{2}(\angle BOD + \angle AOC)$.

16. AB is the common chord of two circles of equal radius. If a line segment meet through the circles at P and Q, prove that ΔOAQ is an isosceles triangle.

17. If the chord $AB = x$ cm and $OD \perp AB$, are in the circle ABC with centre O use the adjoint figure to answer the following questions:



a. Find the area of the circle.

b. Show that D is the mid point of AB.

c. If $OD = (\frac{x}{2} - 2)$ cm, determine x.

18. The lengths of three sides of a triangle are 4 cm, 5 cm and 6 cm respectively. Use this information to answer the following questions:

a. Construct the triangle.

b. Draw the circumcircle of the triangle.

c. From an exterior point of the circumcircle,, draw two tangents to it and show that their lengths are equal.

Chapter Nine

Trigonometric Ratio

In our day to day life we make use of triangles, and in particular, right angled triangles. Many different examples from our surroundings can be drawn where right triangles can be imagined to be formed. In ancient times, with the help of geometry men learnt the technique of determining the width of a river by standing on its bank. Without climbing the tree they knew how to measure the height of the tree accuratly by comparing its shadow with that of a stick. In all the situations given above, the disttances or heights can be found by using some mathematical technique which come under a special branch of mathematics called Trigonometry. The word 'Trigonometry' is derived from Greek words 'tri' (means three), 'gon' (means edge) and `metron' (means measure). In fact, trigonometry is the study of relationship between the sides and angles of a triangle. There are evidence of using the Trigonometry in Egyptian and Babilian civilization. It is believed that the Egyptians made its extensive use in land survey and engineering works. Early astrologer used it to determine the distances from the Earth to the farest planets and stars. At present trigonometry is in use in all branches of mathematics. There are wide usages of trigonometry for the solution of triangle related problems and in navigation etc. Now a days trigonometry is in wide use in Astronomy and Calculus.

At the end of the chapter, the students will be able to –

- Describe the trigonometric ratios of acute angles
- Determine the mutual relations among the trigonometric ratios of acute angle
- Solve and prove the mathematical problems justifying the trigonometric ratios of acute angle
- Determine and apply trigonometric ratios of acute angles 30^o. 45^o, 60^o geometrically
- Determine and apply the value of meaningful trigonometric ratios of the angles 0^o and 90^o
- Prove the trigonometric identities
- Apply the trigonometric identities.

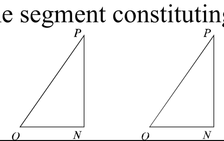
9·1 Naming of sides of a right angled triangle

We know that, the sides of right angles triangle are known as hypotenuse, base and height. This is successful for the horizontal position of triangle. Again, the naming of sides is based on the position of one of the two acute angles of right angled triangle. As for example :

a. 'Hypotenuse', the side of a right angled triangle, which is the opposite side of the right angle.

b. 'Opposite side', which is the direct opposite side of a given angle.

c. 'Adjacent side,' which is a line segment constituting the given angle.



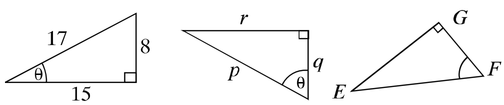
For the angle $\angle PON$, OP is the hypotenuse, ON is the adjacent side and PN is the opposite side.	For the angle $\angle OPN$, OP is the hypotenuse, PN is the adjacent side and ON is the opposite side.

In the geometric figure, the capital letters are used to indicate the vertices and small letters are used to indicate the sides of a triangle. We often use the Greek letters to indicate angle. Widely used six letters of Greek alphabet are :

alpha α	beta β	gamma γ	theta θ	phi ϕ	omega ω

Greek letter are used in geometry and trigonometry through all the great mathematician of ancient Greek.

Example 1. Indicate the hypotenuse, the adjacent side and the opposite side for the angle θ .



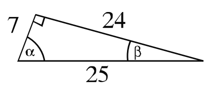
Solution:

(a) hypotenuse 17 units
opposite side 8 units
adjacent side 15 units

(b) hypotenuse p
opposite side r
adjacent side q

(c) hypotenuse EF
opposite side EG
adjacent side FG

Example 2. Find the lengths of hypotenuse, the adjacent side and the opposite side for the angles α and β .

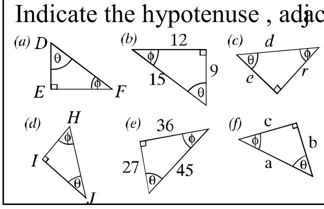


(a) For α angle,
hypotenuse 25 units
opposite side 24 units
adjacent side 7 units.

(b) For β angle
hypotenuse 25 units
opposite side 7 units
adjacent side 24 units.

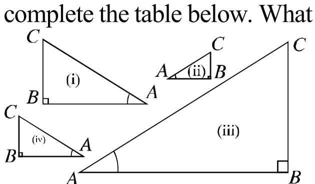
Activity :

Indicate the hypotenuse , adjacent side and opposite for the angle θ and ϕ .



9·2 Constantness of ratios of the sides of similar right-angled triangles

Activity : Measure the lengths of the sides of the following four similar triangles and complete the table below. What do you observe about the ratios of the triangles ?



length of sides			ratio (related to angle)		
BC	AB	AC	BC/AC	AB/AC	BC/AB

Let, $\angle XOA$ is an acute angle. We take a point P on the side OA. We draw a perpendicular from P to OX. As a result, a right angled triangle POM is formed. The three ratios of the sides PM, OM and OP of ΔPOM do not depend on the position of the point P on the side OA.

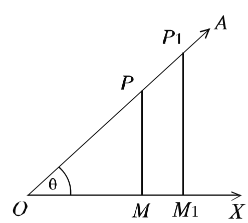
If we draw the perpendiculars PM and P_1M_1 from two points P and P_1 of OA to the side OX, two similar triangles ΔPOM and ΔP_1OM_1 are formed.

Now, ΔPOM and ΔP_1OM_1 are being similar,

$$\frac{PM}{P_1M_1} = \frac{OP}{OP_1} \text{ or, } \frac{PM}{OP} = \frac{P_1M_1}{OP_1} \quad ..... (i)$$

$$\frac{OM}{OM_1} = \frac{OP}{OP_1} \text{ or, } \frac{OM}{OP} = \frac{OM_1}{OP_1} \quad .... (ii)$$

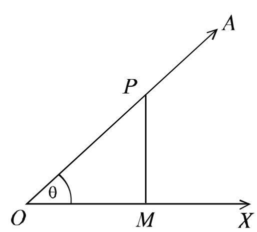
$$\frac{PM}{P_1M_1} = \frac{OM}{OM_1} \text{ or, } \frac{PM}{OM} = \frac{P_1M_1}{OM_1} \quad .... (iii)$$



That is, each of these ratios is constant. These ratios are called trigonometric ratios.

9·3 Trigonometric ratios of an acute angle

Let, $\angle XOA$ is an acute angle. We take any point P on OA. We draw a perpendicular PM from the point P to OX. So, a right angled triangle POM is formed. The six ratios are obtained from the sides PM, OM and OP of ΔPOM which are called trigonometric ratios of the angle $\angle XOA$ and each of them are named particularly.



With respect to the $\angle XOA$ of right angled triangle POM, PM is the opposite side. OM is the adjacent side and OP is the hypotenuse. Denoting $\angle XOA = \theta$, the obtained six ratios are described below for the angle

From the figure :

$\sin\theta = \frac{PM}{OP} = \frac{\text{opposite side}}{\text{hypotenuse}}$ [sine of angle θ]

$\cos\theta = \frac{OM}{OP} = \frac{\text{adjacent side}}{\text{hypotenuse}}$ [cos*ine* of angle θ]

$\tan\theta = \frac{PM}{OM} = \frac{\text{opposite}}{\text{sideadjacent side}}$ [tan *gent* of angle θ]

And opposite ratios of them are

$\text{cosec}\theta = \frac{1}{\sin\theta}$ [cosec*ant* of angle θ]

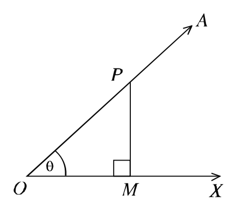
$\sec\theta = \frac{1}{\cos\theta}$ [sec *ant* of angle θ]

$\cot\theta = \frac{1}{\tan\theta}$ [cot *angent* of angle θ]

We observe, the symbol $\sin\theta$ means the ratio of sine of the angle θ, not the multiplication of sin and θ. sin is meaningless without θ. It is applicable for the other trigonometric ratios as well.

9·4 Relation among the trigonometric ratios

Let, $\angle XOA = \theta$ is an acute angle.
from the adjacent figure, according to the definition



$\sin\theta = \frac{PM}{OP}$, $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{OP}{PM}$

$\cos\theta = \frac{OM}{OP}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{OP}{OM}$

$\tan\theta = \frac{PM}{OM}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{OM}{PM}$

Again, $\tan\theta = \frac{PM}{OM} = \frac{\frac{PM}{OP}}{\frac{OM}{OP}}$ [Dividing the numerator and the denominator by OP]

$= \frac{\sin\theta}{\cos\theta}$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

and similarly

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

9·5 Trigonometric identity

(i) $(\sin\theta)^2+(\cos\theta)^2 = \left(\frac{PM}{OP}\right)^2+\left(\frac{OM}{OP}\right)^2$

$= \frac{PM^2}{OP^2}+\frac{OM^2}{OP^2} = \frac{PM^2+OM^2}{OP^2} = \frac{OP^2}{OP^2}$ [by the formula of Pythagoras]

$=1$

or, $(\sin\theta)^2+(\cos\theta)^2=1$

$\therefore\ \sin^2\theta+\cos^2\theta=1$

Remark : For integer indices n we can write $\sin^n\theta$ for $(\sin\theta)^n$ and $\cos^n\theta$ for $(\cos\theta)^n$.

(ii) $\sec^2\theta=(\sec\theta)^2 = \left(\frac{OP}{OM}\right)^2$

$= \frac{OP^2}{OM^2} = \frac{PM^2+OM^2}{OM^2}$ [OP is the hypotenuse of right angled ΔPOM]

$= \frac{PM^2}{OM^2}+\frac{OM^2}{OM^2}$

$= 1+\left(\frac{PM}{OM}\right)^2 = 1+(\tan\theta)^2 = 1+\tan^2\theta$

$\therefore\ \sec^2\theta=1+\tan^2\theta$

or, $\sec^2\theta-\tan^2\theta=1$

or, $\tan^2\theta=\sec^2\theta-1$

(iii) $\text{cosec}^2\theta=(\text{cosec}\,\theta)^2 = \left(\frac{OP}{PM}\right)^2$

$= \frac{OP^2}{PM^2} = \frac{PM^2+OM^2}{PM^2}$ [is the hypotenuse of rightangled ΔPOM]

$= \frac{PM^2}{PM^2}+\frac{OM^2}{PM^2} = 1+\left(\frac{OM}{PM}\right)^2$

$= 1+(\cot\theta)^2 = 1+\cot^2\theta$

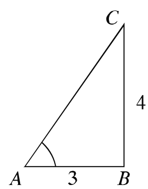
$\therefore$ $\text{cosec}^2\theta-\cot^2\theta=1$ and $\cot^2\theta=\text{cosec}^2\theta-1$

Activity :

1. Construct a table of the following trigonometric formulae for easy memorizing.

$\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	$\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ $\text{cosec}^2\theta = 1 + \cot^2\theta$

Example 1. If $\tan A = \frac{4}{3}$, find the other trigonometric ratios of the angle A.



Solution : Given that, $\tan A = \frac{4}{3}$.

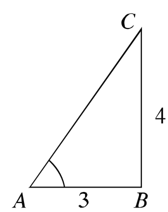
So, opposite side of the angle $A = 4$, adjacent side $= 3$

hypotenuse $= \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Therefore,, $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\cot A = \frac{3}{4}$

$\text{cosec}\, A = \frac{5}{4}$, $\sec A = \frac{5}{3}$.

Example 2. $\angle B$ is the right angle of a right angled triangle ABC. If $\tan A = \frac{3}{4}$, verify the truth of $2\sin A\cos A = 1$.



Solution : Given that, $\tan A = \frac{3}{4}$,

So, opposite side of the angle $A = 4$, adjacent side $= 3$

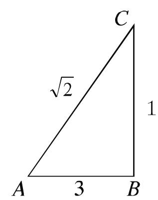
hypotenuse $= \sqrt{4^2 + 3^2} = \sqrt{5} = 5$

so, $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{4}$

Hence, $2\sin A\cos A = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25} \neq 1$

Therefore, $2\sin A\cos A = 1$ is a false statement.

Example 3. $\angle B$ is a right angle of a right angled triangle ABC. If $\tan A = 1$, verify the justification of $2\sin A\cos A = 1$.



Solution : Given that, $\tan A = 1$.
So, opposite side of the $A = 1$, adjacent side $= 1$

hypotenuse = $\sqrt{1^2 + 1^2} = \sqrt{2}$

Therefore, $\sin A = \dfrac{1}{\sqrt{2}}$, $\cos A = \dfrac{1}{\sqrt{2}}$.

Now left hand side $= 2 \sin A \cos A = 2 \cdot \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} = 2 \cdot \dfrac{1}{2} = 1$

= right hand side.

$\therefore\ 2\sin A \cos A = 1$ is a true statement.

> **Activity :**
> 1. If $\angle C$ is a right angle of a right angled triangle ABC, $AB = 29$ cm, $BC = 21$ cm and $\angle ABC = \theta$, find the value of $\cos^2\theta - \sin^2\theta$.

Example 4. Prove that, $\tan\theta + \cot\theta = \sec\theta, \cos ec\theta$.

Proof :

Left hand side = $\tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cdot \cos\theta}$$

$$= \frac{1}{\sin\theta \cdot \cos\theta} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$$

$= \text{cosec}\theta \cdot \sec\theta$

$= \sec\theta \cdot \text{cosec}\theta =$

= R.H.S. (proved)

Example 5. Prove that, $\dfrac{1}{1 + \sin^2\theta} + \dfrac{1}{1 + \cos ec^2\theta} = 1$

Proof : L.H.S. = $\dfrac{1}{1 + \sin^2\theta} + \dfrac{1}{1 + \text{cosec}^2\theta}$

$$= \frac{1}{1 + \sin^2\theta} + \frac{1}{1 + \dfrac{1}{\sin^2\theta}}$$

$$= \frac{1}{1 + \sin^2\theta} + \frac{\sin^2\theta}{1 + \sin^2\theta}$$

$= \dfrac{1+\sin^2\theta}{1+\sin^2\theta}$

$= 1 =$ R.H.S. (proved)

Example 6. Prove that : $\dfrac{1}{2-\sin^2 A}+\dfrac{1}{2+\tan^2 A}=1$

Proof : L.H.S. $= \dfrac{1}{2-\sin^2 A}+\dfrac{1}{2+\tan^2 A}$

$= \dfrac{1}{2-\sin^2 A}+\dfrac{1}{2+\dfrac{\sin^2 A}{\cos^2 A}}$

$= \dfrac{1}{2-\sin^2 A}+\dfrac{\cos^2 A}{2\cos^2 A+\sin^2 A}$

$= \dfrac{1}{2-\sin^2 A}+\dfrac{\cos^2 A}{2(1-\sin^2 A)+\sin^2 A}$

$= \dfrac{1}{2-\sin^2 A}+\dfrac{\cos^2 A}{2-2\sin^2 A+\sin^2 A}$

$= \dfrac{1}{2-\sin^2 A}+\dfrac{1-\sin^2 A}{2-\sin^2 A}$

$= \dfrac{2-\sin^2 A}{2-\sin^2 A}$

$= 1 =$ R.H.S. (proved)

Example 7. Prove that : $\dfrac{\tan A}{\sec A+1}-\dfrac{\sec A-1}{\tan A}=0$

proof : L.H.S. $= \dfrac{\tan A}{\sec A+1}-\dfrac{\sec A-1}{\tan A}$

$= \dfrac{\tan^2 A-(\sec^2 A-1)}{(\sec A+1)\tan A}$ $[\sec^2\theta-1=\tan^2\theta]$

$= \dfrac{\tan^2 A-\tan^2 A}{(\sec A+1)\tan A}$

$= \dfrac{0}{(\sec A+1)\tan A}$

$= 0 =$ R.H.S. (proved)

Example 8. Prove that : $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$

Prove : L.H.S. = $\sqrt{\dfrac{1-\sin A}{1+\sin A}}$

$= \sqrt{\dfrac{(1-\sin A)(1-\sin A)}{(1+\sin A)(1-\sin A)}}$ [Multiplying the numerator and the denominator by $\sqrt{(1-\sin A)}$]

$= \sqrt{\dfrac{(1-\sin A)^2}{1-\sin^2 A}}$

$= \sqrt{\dfrac{(1-\sin A)^2}{\cos^2 A}}$

$= \dfrac{1-\sin A}{\cos A}$

$= \dfrac{1}{\cos A} - \dfrac{\sin A}{\cos A}$

$= \sec A - \tan A$

= R.H.S. (proved).

Example 9. If $\tan A + \sin A = a$ and $\tan A - \sin A = b$, prove that, $a^2 - b^2 = 4\sqrt{ab}$.

Prove : We given that, $\tan A + \sin A = a$ and $\tan A - \sin A = b$

L.H.S. = $a^2 - b^2$

$= (\tan A + \sin A)^2 - (\tan A - \sin A)^2$

$= 4\tan A \sin A$ $[\because (a-b)^2 - (a+b)^2 = 4ab]$

$= 4\sqrt{\tan^2 A \sin^2 A}$

$= 4\sqrt{\tan^2 A (1-\cos^2 A)}$

$= 4\sqrt{\tan^2 A - \tan^2 A \cdot \cos^2 A}$

$= 4\sqrt{\tan^2 A - \sin^2 A}$

$= 4\sqrt{(\tan A + \sin A)(\tan A - \sin A)}$

$= 4\sqrt{ab}$

= R.H.S. (proved)

> **Activity :** 1. If $\cot^4 A - \cot^2 A = 1$, prove that, $\cos^4 \theta + \cos^2 A = 1$
> 2. If $\sin^2 A - \sin^4 A = 1$, prove that, $\tan^4 A + \tan^2 A = 1$

Example 10. If $\sec A + \tan A = \dfrac{5}{2}$, find the value of $\sec A - \tan A$.

Solution: We given that, $\sec A + \tan A = \dfrac{5}{2}$ ....(i)

We know that, $\sec^2 A = 1 + \tan^2 A$

or, $sec^2 A - tan^2 A = 1$

or, $(secA + tanA)(secA - tanA) = 1$

or, $\frac{5}{2}(secA - tanA) = 1$ [from (i)]

$\therefore \quad secA - tanA = \frac{2}{5}$

Exercise 9·1

1. Verify whether each of the following mathematical statements is true or false. Give argument in favour of your answer.
   (a) The value of $\tan A$ is always less than 1.
   (b) $\cot A$ is the multiplication of cot and A.
   (c) For any value of A, $\sec A = \frac{12}{5}$.
   (d) cos is the smallest form of cotangent.
2. If $\sin A = \frac{3}{4}$, find the other trigonometric ratios of the angle A.
3. Given that $15\cot A = 8$, find the values of $\sin A$ and $\sec A$.
4. If $\angle C$ is the right angle of the right angled triangle ABC, $AB = 13$ cm and $BC = 12$ cm. and $\angle ABC = \theta$, find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.
5. $\angle B$ is the right angle of the right angled triangle ABC. If $\tan A = \sqrt{3}$, verify the truth of $\sqrt{3}\sin A\cos A = 4$.

Prove (6 – 20) :

6. (i) $\frac{1}{\sec^2 A} + \frac{1}{\text{cosec}^2 A} = 1$; (ii) $\frac{1}{\cos^2 A} - \frac{1}{\cot^2 A} = 1$; (iii) $\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} = 1$;
7. (i) $\frac{\sin A}{\text{cosec} A} + \frac{\cos A}{\sec A} = 1$; (ii) $\frac{\sec A}{\cos A} - \frac{\tan A}{\cot A} = 1$. (iii) $\frac{1}{1+\sin^2 A} + \frac{1}{1+\text{cosec}^2 A} = 1$
8. (i) $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = secA \cdot cosecA + 1$; (ii) $\frac{1}{1+\tan^2 A} + \frac{1}{1+\cot^2 A} = 1$
9. $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = sinA + \cos A$.
10. $\tan A\sqrt{1`-\sin^2 A} = \sin A$.
11. $\frac{\sec A + \tan A}{\text{cosec} A + \cot A} = \frac{\text{cosec} A - \cot A}{\sec A - \tan A}$
12. $\frac{\text{cosec} A}{\text{cosec} A - 1} + \frac{\text{cosec} A}{\text{cosec} A + 1} = 2\sec^2 A$.
13. $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$.
14. $\frac{1}{\text{cosec} A - 1} - \frac{1}{\text{cosec} A + 1} = 2\tan^2 A$.
15. $\frac{\sin A}{1-\cos A} + \frac{1-\cos A}{\sin A} = 2\text{ cosec } A$.
16. $\frac{\tan A}{\sec A + 1} - \frac{\sec A - 1}{\tan A} = 0$

17. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$ 18. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \,.\, \tan B.$

19. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$ 20. $\sqrt{\dfrac{\sec A+1}{\sec A-1}} = \cot A + \text{cosec } A.$

21. If $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \sqrt{2}\sin A$

22. If $\tan A = \dfrac{1}{\sqrt{3}}$, find the value of $\dfrac{\text{cosec}^2 A - \sec^2 A}{\text{cosec}^2 A + \sec^2 A}$.

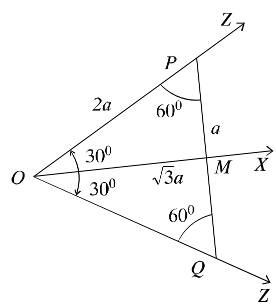
23. If $\text{cosec} A - \cot A = \dfrac{4}{3}$, what is the value of $\text{cosec} A + \cot A$?

24. If $\cot A = \dfrac{b}{a}$, find the value of $\dfrac{a \sin A - b \cos A}{a \sin A + b \cos A}$.

9·6 Trigonometric ratios of the angles 30°, 45° and 60°

We have learnt to draw the angles having the measurement of 30°, 45° and 60° geometrically. The actual values of the trigonometric ratios for all these angles can be determined geometrically.

Trigonometric ratios of the angles 30° and 60°



Let, $\angle XOZ = 30^\circ$ and P is a point on the side OZ. Draw $PM \perp OX$ and extend PM upto Q such that $MQ = PM$. Add O, Q and extend upto Z.

Now, between ΔPOM and ΔQOM, $PM = QM$, OM is the common side and included $\angle PMO$ = included $\angle QMO = 90^\circ$

$\therefore \; \Delta POM \cong \Delta QOM$

Therefore, $\angle QOM = \angle POM = 30^\circ$

and $\angle OQM = \angle OPM = 60^\circ$

Again, $\angle POQ = \angle POM + \angle QOM = 30^\circ + 30^\circ = 60^\circ$

$\therefore \; \Delta OPQ$ is an equilateral triangle.

If $OP = 2a$, $PM = \dfrac{1}{2}PQ = \dfrac{1}{2}OP = a$ [since ΔPOQ is an equilateral triangle]

From right-angled ΔOPM, we get,

$OM = \sqrt{OP^2 - PM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$.

We find the trigonometric ratios :

$\therefore \; \sin 30^\circ = \dfrac{PM}{OP} = \dfrac{a}{2a} = \dfrac{1}{2}$, $\cos 30^\circ = \dfrac{OM}{OP} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$

$$\tan 30^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}.$$

$$\text{cosec } 30^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2, \ \sec 30^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}.$$

Similarly,

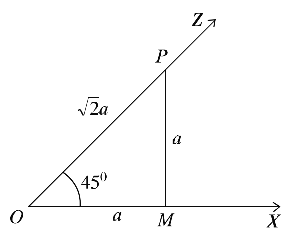
$$\sin 60^\circ = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{PM}{OP} = \frac{a}{2a} = \frac{1}{2}, \tan 60^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\text{cosec } 60^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}},$$

$$\sec 60^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2, \cot 60^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}.$$

Trigonometric ratio of the angle 45°



Let, $\angle XOZ = 45^\circ$ and P is a point on OZ.
Draw $PM \perp OX$. In right angled triangle ΔOPM, $\angle POM = 45^\circ$

So, $\angle OPM = 45^\circ$

Therefore, $PM = OM = a$ (suppose)

Now, $OP^2 = OM^2 + PM^2 = a^2 + a^2 = 2a^2$

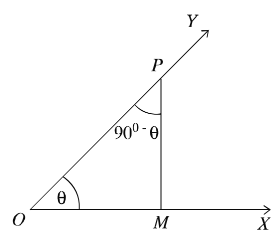
or, $OP = \sqrt{2}\,a$

From the definition of trigonometric ratios, we get

$$\sin 45^\circ = \frac{PM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \ \cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \ \tan 45^\circ = \frac{PM}{OM} = \frac{a}{a} = 1$$

$$\text{cosec } 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}, \ \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}, \ \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

9·7 Trigonometric ratios of complementary angles



We know, if the sum of two acute angles is 90°, one of them is called complementary angle to the other. For example, 30° and 60°; 15° and 75° are complementary angles to each other.

In general, the angles θ and $(90^\circ - \theta)$ are complementary angles to each other.

Trigonometric ratios of complementary angles

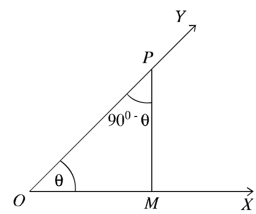
Let, $\angle XOY = \theta$ and P is the point on the side OY of the angle. We draw $PM \perp OX$.

Since the sum of the three angles of a triangle is two right angles therefore, in the right angled triangle POM, $\angle PMO = 90^\circ$

and $\angle OPM + \angle POM =$ one right angle $= 90^\circ$

$\therefore \ \angle OPM = 90^\circ - \angle POM = 90^\circ - \theta$

[Since $\angle POM = \angle XOY = \theta$]



$\therefore \quad \sin\ (90^\circ - \theta) = \dfrac{OM}{OP} = \cos \angle POM = \cos\theta$

$\cos (90^\circ - \theta) = \dfrac{PM}{OP} = \sin \angle POM = \sin\theta$

$\tan\ (90^\circ - \theta) = \dfrac{OM}{PM} = \cot \angle POM = \cot\ \theta$

$\cot\ (90^\circ - \theta) = \dfrac{PM}{OM} = \tan \angle POM = \tan\ \theta$

$\sec (90^\circ - \theta) = \dfrac{OP}{PM} = \text{cosec} \angle POM = \text{cosec}\ \theta$

$\text{cosec} (90^\circ - \theta) = \dfrac{OP}{OM} = \sec \angle POM = \sec\ \theta$.

We can express the above formulae in words below :

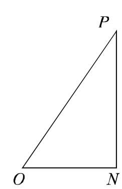
sine of complementary angle = *cosine* of angle

cosine of complementary angle = *sine* of angle

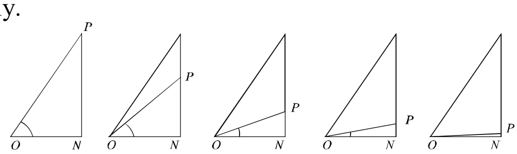
tangent of complementary angle = *cotangent* of angle etc.

Activity : 1. If $\sec (90^\circ - \theta) = \dfrac{5}{3}$**,** find the value of $\text{cosec}\theta - \cot\theta$ **.**

9·8 Trigonometric ratios of the angles 0° and 90°



We have learnt how to determinal the trigonometric ratios for the acute angle θ of a right angled triangle. Now, we see, if the angle is made gradually smaller, how the trigonometric ratios change. As θ get smallers the length of the side PN also gets smaller. The point P closes to the point N and finally the angle θ comes closer to the angle 0°, OP is reconciled with ON approximately.



When the angle θ comes closer to 0°, the length of the line segment PN reduces to zero and in this case the value of $\sin\theta \frac{PN}{OP}$ is approximately zero. At the same time, the length of OP is equal to the length of ON and the value of $\cos\theta = \frac{ON}{OP}$ is 1 approximately.

The angle, 0° is introduced for the convenience of discussion in trigonometry, and the edge line and the original line of the angle 0° are supposed the same ray. Therefore, in line with the prior discussion, it is said that, $\cos 0^\circ = 1$, $\sin 0^\circ = 0$.

If θ is the acute angle, we see

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

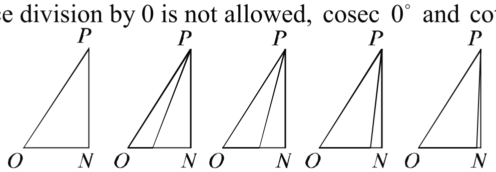
$$\sec\theta = \frac{1}{\cos\theta}, \quad \text{cosec}\theta = \frac{1}{\sin\theta},$$

We define the angle 0° in probable cases so that, those relations exists.

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1.$$

Since division by 0 is not allowed, cosec 0° and $\cot 0^\circ$ can not be defined.



Again, when the angle θ is very closed to 90°, hypotenuse OP is approximatel y equal to PN. So the value of $\sin\theta$ is approximately 1. On the other hand, if the angle θ is is equal to 90°, ON is nearly zero;the value of $\cos\theta$ is approximately 0. So, in agreement of formulae that are described above, we can say, $\cos 90^\circ = 0$, $\sin 90^\circ = 1$.

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

$$\text{cosec}\, 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1$$

Since one can not divided by 0, as before, $\tan 90^\circ$ and $\sec 90^\circ$ are not defined.

Observe : For convenience of using the values of trigonometric ratios of the angles 0°, 30°, 45°, 60° and 90° are shown in the following table :

angle / Ratio	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cosecant	undefined	$\frac{1}{2}$	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Observe : Easy method for remembering of the values of trigonometric ratios of some fixed angles.

(*i*) If we divide the numbers $0, 1, 2, 3$ and 4 by 4 and take square root of the quotients, we get the values of $\sin 0^\circ$ of $\sin 30^\circ$, $\sin 45^\circ$, $\sin 60^\circ$ and $\sin 90^\circ$ respectively.

(*ii*) If we divide the numbers $4, 3, 2, 1$ and 0 by 4 and take square root of quotients, we get the values of $\cos 0^\circ$, $\cos 30^\circ$, $\cos 45^\circ$, $\cos 60^\circ$ and $\cos 90^\circ$ respectively.

(*iii*) If we divide the numbers $0, 1, 3$ and 9 by 3 and take square root of quotients, we get the values of $\tan 0^\circ$, $\tan 30^\circ$, $\tan 45^\circ$ and $\tan 60^\circ$, respectively (It is noted that tan 90° is undefined).

(*iv*) If we divide the numbers $9, 3, 1$ and 0 by 3 and take square root of quotients, we get the values of $\cot 30^\circ$, $\cot 45^\circ$, $\cot 60^\circ$, $\cot 90^\circ$ respectively (It is noted that $\cot 0^\circ$ is undefined).

Example 1. Find the values :

(a) $\dfrac{1 - \sin^2 45^\circ}{1 + \sin^2 45^\circ} + \tan^2 45^\circ$

(b) $\cot 90^\circ \cdot \tan 0^\circ \cdot \sec 30^\circ \cdot \operatorname{cosec} 60^\circ$

(c) $\sin 60^\circ \cdot \cos 30^\circ + \cos 60^\circ \cdot \sin 30^\circ$

(d) $\dfrac{1 - \tan^2 60^\circ}{1 + \sin^2 60^\circ} + \sin^2 60^\circ$

Solution :

(a) Given expression = $\dfrac{1-sin^2 45^\circ}{1+sin^2 45^\circ}+tan^2 45^\circ$

$$=\frac{1-\left(\frac{1}{\sqrt{2}}\right)^2}{1+\left(\frac{1}{\sqrt{2}}\right)^2}+(1)^2 \quad [\because sin 45^\circ=\frac{1}{\sqrt{2}} \mid tan 45^\circ=1]$$

$$=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}+1=\frac{\frac{1}{2}}{\frac{3}{2}}+1=\frac{1}{3}+1=\frac{4}{3}$$

(b) Given expression = $cot\,90^\circ\cdot tan\,0^\circ\cdot sec\,30^\circ\cdot cosec\,60^\circ$

$$=0\cdot 0\cdot\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}=0$$

$$[\because cot\,90^\circ=0,\ tan\,0^\circ=0,\ sec\,30^\circ=\frac{2}{\sqrt{3}},\ cosec\,60^\circ=\frac{2}{\sqrt{3}}\ ]$$

(c) Given expression = $sin\,60^\circ\cdot cos\,30^\circ+cos\,60^\circ\cdot sin\,30^\circ$

$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot\frac{1}{2}$$

$$[\because \sin 60^\circ=\cos 30^\circ=\frac{\sqrt{3}}{2},\ \sin 30^\circ=\cos 60^\circ=\frac{1}{2}\ ]$$

$$=\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1$$

(d) Given expression = $\dfrac{1-\tan^2 60^\circ}{1+\sin^2 60^\circ}+\sin^2 60^\circ$

$$=\frac{1-(\sqrt{3})^2}{1+(\sqrt{3})^2}+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$=\frac{1-3}{1+3}+\frac{3}{4}=\frac{-2}{4}+\frac{3}{4}$$

$$=\frac{-2+3}{4}=\frac{1}{4}$$

Example 2.

(a) If $\sqrt{2}\cos(\mathrm{A}-\mathrm{B})=1,\ 2\sin(\mathrm{A}+\mathrm{B})=\sqrt{3}$ and A, B are acute angles, find the values of A and B.

(b) If $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$, find the value of A.

(c) Prove that, $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$, if $A=45^\circ$.

(d) Solve : $2\cos^2\theta+3\sin\theta-3=0$, where θ is an acute angle.

Solution : (a) $\sqrt{2}cos(A-B)=1$

or, $cos(A-B)=\dfrac{1}{\sqrt{2}}$

or, $cos(A-B)=cos45^\circ$ $[\because cos45^\circ=\dfrac{1}{\sqrt{2}}]$

$\therefore$ $A-B=45^\circ$..................(i)

and $2sin(A+B)=\sqrt{3}$

or, $sin(A+B)=\dfrac{\sqrt{3}}{2}$

or, $sin(A+B)=sin\,60^\circ$ $[\because sin\,60^\circ=\dfrac{\sqrt{3}}{2}]$

$\therefore$ $A+B=60^\circ$....................(ii)

Adding (i) and (ii), we get,

$2A=105^\circ$

$\therefore$ $A=\dfrac{105^\circ}{2}=52\dfrac{1}{2}^\circ$

Again, subtracting (i) from (ii), we get

$2B=15^\circ$

or, $B=\dfrac{15^\circ}{2}$

$\therefore$ $B=7\dfrac{1}{2}^\circ$

Required $A=52\dfrac{1}{2}^\circ$ and $B=7\dfrac{1}{2}^\circ$

(b) $\dfrac{cosA-sinA}{cosA+sinA}=\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$

or, $\dfrac{cosA-sinA+cosA-sinA}{cosA-sinA-cosA-sinA}=\dfrac{1-\sqrt{3}+1-\sqrt{3}}{1-\sqrt{3}+1-\sqrt{3}}$

or, $\dfrac{2\cos A}{-2\sin A} = \dfrac{2}{-2\sqrt{3}}$

or, $\dfrac{\cos A}{\sin A} = \dfrac{1}{\sqrt{3}}$

or, $\cot A = \cot 60^\circ$

$\therefore$ $A = 60^\circ$

(c) Given that, $A = 45^\circ$

we have to prove that, $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

L.H.S. = $\cos 2A$

$= \cos(2\times 45^\circ) = \cos 90^\circ = 0$

R.H.S. = $\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$= \dfrac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ} = \dfrac{1-(1)^2}{1+(1)^2}$

$= \dfrac{0}{2} = 0$

$\therefore$ L.H.S. = R.H.S. (proved)

(d) Given equation, $2\cos^2\theta + 3\sin\theta - 3 = 0$

or, $2(1-\sin^2\theta) - 3(1-\sin\theta) = 0$

or, $2(1+\sin\theta)(1-\sin\theta) - 3(1-\sin\theta) = 0$

or, $(1-\sin\theta)\{2(1+\sin\theta) - 3\} = 0$

or, $(1-\sin\theta)\{2\sin\theta - 1\} = 0$

or, $1-\sin\theta = 0$ or $2\sin\theta - 1 = 1$

$\therefore$ $\sin\theta = 1$ or, $2\sin\theta = 1$

or, $\sin\theta = \sin 90^\circ$ or, $\sin\theta = \dfrac{1}{2}$

$\therefore$ $\theta = 90^\circ$ or, $\sin\theta = \sin 30^\circ$

or, $\theta = 30^\circ$

θ is an acute angle, so $\theta = 30^\circ$.

Exercise 9·2

1. If $\cot\theta = \dfrac{1}{2}$, which one is the value of $\cot\theta$?

(a) $\dfrac{1}{\sqrt{3}}$ (b) 1 (c) $\sqrt{3}$ (d) 2

2.

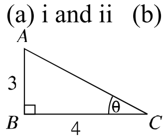
(i) $\sin^2\theta = 1-\cos^2\theta$

(ii) $\sec^2\theta = 1+\tan^2\theta$

(iii) $\cot^2\theta = 1-\tan^2\theta$

Which one of the followings is correct in accordance with the above statements.

(a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii



Answer of questions 3 and 4 on the basis of the figure :

3. What is the value of $\sin\theta$?

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

4. What is the value of $\cot\theta$?

(a) $\frac{3}{4}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{4}{3}$

Evaluate (5 – 8) :

5. $\frac{1-\cot^2 60^\circ}{1-\cot^2 60^\circ}$.

6. $\tan 45^\circ \cdot \sin^2 60^\circ \cdot \tan 30^\circ \cdot \tan 60^\circ$.

7. $\frac{1-\cos^2 60^\circ}{1-\cos^2 60^\circ} + \sec^2 60^\circ$.

8. $\cos 45^\circ \cdot \cot^2 60^\circ \cdot \operatorname{cosec}^2 30^\circ$.

Prove (9 – 11) :

9. $\cos^2 30^\circ - \sin^2 30^\circ = \cos 60^\circ$.

10. $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ = \sin 90^\circ$.

11. $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \cos 30^\circ$.

12. $\sin 3A = \cos 3A$, if $A = 15^\circ$.

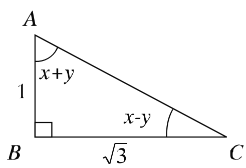
13. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, if $A = 45^\circ$.

14. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$, if $A = 30^\circ$.

15. $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$, if $A = 60^\circ$.

16. If $2\cos(A+B) = 1 = 2\sin(A-B)$ and A, B are acute angles, show that $A = 45^\circ$, $B = 15^\circ$.

17. If $\cos(A - B) = 1$, $2\sin(A + B)$ and A, B are acute angle, find the values of A and B.
18. Solve : $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.
19. If A, B are acute angle and $\cot(A + B) = 1$, $\cot(A - B) = \sqrt{3}$, find the values of A and B.
20. Show that, $\cos 3A = 4\cos^3 A - 3\cos A$, when $A = 30^\circ$.
21. Solve : $\sin\theta + \cos\theta = 1$, when $0^\circ \le \theta \le 90^\circ$.
22. Solve : $\cos^2\theta - \sin^2\theta = 2 - 5\cos\theta$, when θ is an acute angle.
23. Solve : $2\sin^2\theta + 3\cos\theta - 3 = 0$, θ is an acute angle.
24. Solve: $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$.
25. Find the value : $3\cot^2 60^\circ + \frac{1}{4}\operatorname{cosec}^2 30^\circ + 5\sin^2 45^\circ - 4\cos^2 60^\circ$.
26. If $\angle B = 90^\circ$, $AB = 5$cm, $BC = 12$cm. of ΔABC
    (a) Find the length of AC.
    (b) If $\angle C = \theta$, find the value of $\sin\theta + \cos\theta$.
    (c) Show that, $\sec^2\theta + \cos^2\theta = \sec^2\theta \cdot \operatorname{cosec}^2\theta$.
27.



(a) what is the measurement of AC.
(b) Find the value of $\tan A + \tan C$.
(c) Find the values of x and y.

Chapter Ten

Distance and Height

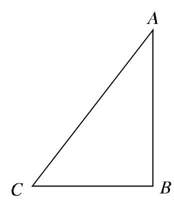
From very ancient times trigonometrical ratios are applied to find the distance and height of any distant object. At present trigonometrcal ratios are of boundless importance because of its increasing usage. The heights of the hills, mountains and trees and the widths of those rivers which cannot be measured in ordinary method are measured the heights and widths with the help of trigonometry. In this condition it is necessary to know the trigonometrical ratios values of acute angle.

At the end of this chapter, the students will be able to –

- Explain the geoline, vertical plane and angles of elevation and declination
- Solve mathematical problem related to distance and height with the help of trigonometry
- Measure practically different types of distances and heights with the help of trigonometry.

Horizontal line, Vertical line and Vertical plane :

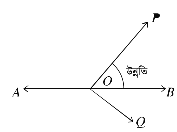
The horizontal line is any straight line on the plane. A straight line parallel to horizon is also called a horizontal line. The vertical line is any line perpendicular to the horizontal plane. It is also called normal line.



A horizontal line and a vertical line intersected at right angles on the plane define a plane. It is known as vertical plane.

In the figure : A tree with height of AB is standing vertically at a distance of CB from a point C on the plane. Here, CB is the horizontal line. BA is the vertical line and the plane ABC is perpendicular to the horizontal plane which is a vertical plane.

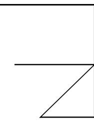
Angle of Elevation and Angle of Depression :



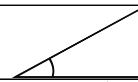
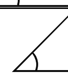
Observe the figure, AB is a straight line parallel to the horizon. The points P, O, B lie on the same vertical plane. The point P on the straight AB makes angle $\angle POB$ with the line AB. Here at O, the angle of elevation of P is $\angle POB$.

So, the angle at any point above the plane with the straight line parallel to horizon is called the angle of elevation.

Again the point Q, O, B lie on the same vertical plane and point Q lines at lower side of the straight line AB parallel to horizon. Here, the angle of depression at O of Q is $\angle QOB$. So, the angle at any point below the straight line parallel to the plane is called the angle of depression.

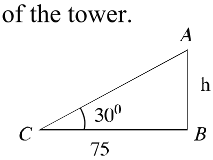
Activity : Point the figure and show the horizontal line, vertical line, vertical plane, angle of elevation and angle of depression.	 ← Land →

N.B. : For solving the problems in this chapter approximately right figure is needed. While drawing the figure, the following techniques are to be applied.

(1) While drawing 30° angle, it is needed base > perpendicular.	
(2) While drawing 45° angle, it is needed base = perpendicular.	
(3) While drawing 60° angle, it is needed base < perpendicular.	

Example 1. The angle of elevation at the top of a tower at a point on the ground is 30° at a distance of 75 metre from the foot. Find the height of the tower.

Solution : Let, the height of the tower is $AB = h$ metre. The angle of elevation at C from the foot of the tower $BC = 75$ metre of A on the ground is $\angle ACB = 30^\circ$



From ΔABC we get, $tan\angle ACB = \dfrac{AB}{BC}$

or, $tan30^\circ = \dfrac{h}{75}$ or, $\dfrac{1}{\sqrt{3}} = \dfrac{h}{75}$ $\left[\because tan30^\circ = \dfrac{1}{\sqrt{3}}\right]$ or, $\sqrt{3}h = 75$ or, $h = \dfrac{75}{\sqrt{3}}$

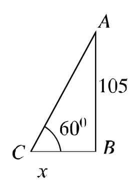
or, $h = \dfrac{75\sqrt{3}}{3}$ [multiplying the numerator and denominator by $\sqrt{3}$] or, $h = 25\sqrt{3}$

$\therefore$ $h = 43.301$ metre (app.).

Required height of the tower is 43.301 metre (app.).

Example 2. The height of a tree is 105 metre. If the angle of elevation of the tree at a point from its foot on the ground is 60°, find the distance of the point on the ground from the foot of the tree.

Solution : Let, the distance of the point on the ground from the foot of tree is $BC = x$ metre. Height of the tree $AB = 105$ metre and at C the angle of elevation of the vertex of tree is $\angle ACB = 60^\circ$



From ΔABC we get,

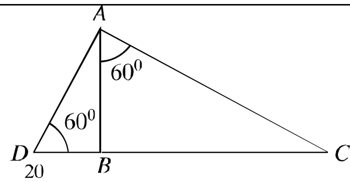
$tan\angle ACB = \frac{AB}{BC}$ or, $tan60^0 = \frac{105}{x}$ $\quad [\because tan60^0 = \sqrt{3}]$

or, $\sqrt{3} = \frac{105}{x}$ or, $\sqrt{3}x = 105$ or, $x = \frac{105}{\sqrt{3}}$ or, $x = \frac{105\sqrt{3}}{3}$ or, $x = 35\sqrt{3}$

$\therefore$ $x = 60.622$ (app.)

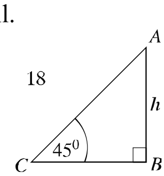
$\therefore$ The required distance of the point on the ground from the foot of the tree is 60.622 metre (app.).

Activity:
In the picture, AB is a tree. Information from the picture –
1. Find the height of the tree.
2. Find the distance of the point C on the ground from the foot of the tree.



Example 3. A ladder of 18 metres. long touches the roof of a wall and makes an angle of 45^o with the horizon. Find the height of the wall.

Solution : Let, the height of the wall AB is $= h$ metre, length of ladder AC is $= 18$ m. and makes angles with the ground $\angle ACB = 45^\circ$.



From ΔABC we get, $\sin\angle ACB = \frac{AB}{AC}$

or, $sin45^\circ = \frac{h}{18}$

or, $\frac{1}{\sqrt{2}} = \frac{h}{18}$ $\quad \left[\because sin45^0 = \frac{1}{\sqrt{2}}\right]$ or, $\sqrt{2}h = 18$ or, $h = \frac{18}{\sqrt{2}}$

or, $\sqrt{2}h = 18$ or, $h = \frac{18}{\sqrt{2}}$

or, $h = \frac{18\sqrt{2}}{2}$ [multiplying the numerator and denominator by $\sqrt{2}$]

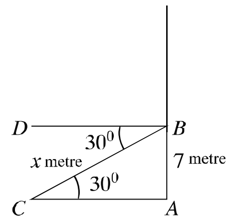
or, $h = 9\sqrt{2}$

$\therefore$ $h = 12.728$ (app.)

Therefore, required height of the wall is 12.728 m. (app.).

Example 4. A tree leaned due to storm. The stick with height of 7 metre from its foot was learned against the tree to make it straight. If the angle of depression at the point of contacting the stick on the ground is 30°, find the length of the stick ?

Solution : Let, the height of the stick from the foot learned against the tree of $AB = 7$ metre and angle of depression is $\angle DBC = 30^\circ$



$\therefore \angle ACB = \angle DBC = 30^\circ$ [alternate angle]

From ΔABC we get,

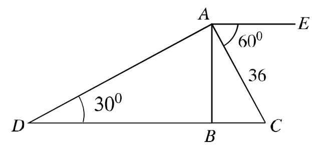
$sin\angle ACB = \dfrac{AB}{BC}$ or, $sin 30^0 = \dfrac{7}{BC}$

or, $\dfrac{1}{2} = \dfrac{7}{BC}$ $\left[\because sin 30^0 = \dfrac{1}{2}\right]$

$\therefore BC = 14$

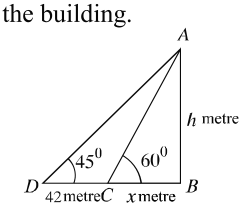
$\therefore$ Required height of the stick is 14 metre.

Activity :
In the figure, if depression angle $\angle CAE = 60^\circ$, elevation angle $\angle ADB = 30^\circ$, $AC = 36$ metre and B, C, D lie on the same straight line, find the lengths of the sides AB, AD and CD.



Example 5. The angle of elevation at a point of the roof of a building is 60° in any point on the ground. Moving back 42 metres from the angle of elevation of the point of the place of the building becomes 45°. Find the height of the building.

Solution : Let, the height of the building is $AB = h$ metres. The angle of elevation at the top $\angle ACB = 60^\circ$. The angle of elevation becomes $\angle ADB = 45^\circ$ moving back from C by $CD = 42$ metres.



Let, $BC = x$ metre

$\therefore BD = BC + CD = (x + 42)$ metre

From ΔABC we get,

$tan 60^0 = \dfrac{AB}{BC}$ or, $\sqrt{3} = \dfrac{h}{x}$ $\left[\because tan 60^0 = \sqrt{3}\right]$

$\therefore x = \dfrac{h}{\sqrt{3}}$............$(i)$

Again, from ΔABD we get, $\tan 45^0 = \dfrac{AB}{BD}$

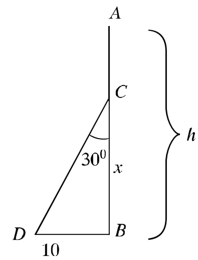
or, $1 = \dfrac{h}{x+42}$ $\left[\because tan 45^0 = 1\right]$ or, $h = x + 42$

or, $h = \dfrac{h}{\sqrt{3}} + 42$; by equation (i) or, $\sqrt{3}h = h + 42\sqrt{3}$ or, $\sqrt{3}h - h = 42\sqrt{3}$ or, $\left(\sqrt{3}-1\right)h = 42\sqrt{3}$ or, $h = \dfrac{42\sqrt{3}}{\sqrt{3}-1}$

$\therefore h = 99.373$ (app.)

Height of the building is 99.373 metres (app.)

Example 6. A pole is broken such that the broken part makes an angle of 30° with the other and touches the ground at a distance of 10 metres form its foot. Find the lengths of the pole.



Solution : Let, the total height of the pole is $AB = h$ metre. Breaks at the height of $BC = x$ matre without separation and makes an angle with the other, $\angle BCD = 30^\circ$ and touches the ground at a distance $BD = 10$ metres from the foot.

Here, $CD = AC = AB - BC = (h - x)$ metre

From ΔBCD we get,

$tan30^\circ = \dfrac{BD}{BC}$ or, $\dfrac{1}{\sqrt{3}} = \dfrac{10}{x}$ $\therefore$ $x = 10\sqrt{3}$

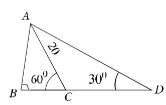
Again, $sin30^\circ = \dfrac{BD}{CD}$ or, $\dfrac{1}{2} = \dfrac{10}{h-x}$

or, $h - x = 20$ or, $h = 20 + x$ or, $h = 20 + 10\sqrt{3}$; [putting the value of x]

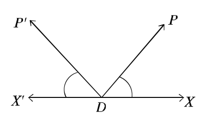
$\therefore h = 37.321$ (app.) $\therefore$ Height of the pole is 37.321 metres (app.).

Activity :
A balloon is flying above any point be tween two mile posts. At the point of the balloon the angle of depression of the two posts are 30° and 60° respectively. Find the height of the balloon.

Exercise 10



1. (a) Find the measurement of $\angle CAD$
   (b) Find the lengths of AB and BC.
   (c) Find the distance between A and D.
2. From a helicopter above a point O between two kilometre posts, the angles of depression of the two points A and B are 60° and 30° respectively.
   (a) Draw a figure with short description.
   (b) Find the height of the helicopter from the ground.
   (c) Find the direct distance from the point A of the helicopter.



3. What is the elevation angle of point P from the point O ?
   (a) $\angle QOB$ (b) $\angle POA$ (c) $\angle QOA$ (d) $\angle POB$
4. (i) The horizontal line is any straight line lying on the plane.
   (ii) Vertical line is any line perpendicular to the plane.
   (iii) A horizontal line and a vertical plane define a plane. It is known as vertical plane.

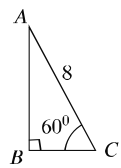
which one is right of the above speech ?

(a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii

Answer the questions 5 – 6 from the adjacent figure :

5. The length of BC will be –

(a) $\frac{4}{\sqrt{3}}$ m (b) 4m (c) $4\sqrt{2}$ m (d) $4\sqrt{3}$ m



6. The length of AB will be –

(a) $\frac{4}{\sqrt{3}}$ m (b) 4m (c) $4\sqrt{2}$ m (d) $4\sqrt{3}$ m

7. If the angle of the elevation of the top of the miner is 30° at a point on the ground and the height is 26 metres, then find the distance of the plane from the Miner.

8. If the top of a tree is 20 metres distance from the foot on the ground at any point and the angle of elevation of 60°, find the height of the tree.

9. Forming 45° angle with ground A 18 metres long ladder touches the top of the wall, find the height of the wall.

10. If the of depression of a point on the ground 20 metres from the top of the house is 30° then, find the height of the house.

11. The angle of elevation of a tower at any point on the ground is 60°. If moved back 25 metre, the angle of elevation becomes 30°, find the height of the tower.

12. The angle of elevation of a tower 60° moving 60 metres towards a minar. Find the height of the minar.

13. A man standing at a place on the bank of a river observesd that the angle of elevation of a tower exactly opposite to him on the other bank was 60^o. Moving 32 metres back he observed that the angle of elevation of the tower was 30^o. Find the height of the tower and the width of the river.

14. A pole of 64 metre long breaks into two parts without complete separation and makes an angle 60° with the ground. Find the length of the broken part of the pole.

15. A tree is broken by a storm such that the broken part makes an angle of 30° with the other and touches the ground at a distance of 10 metres from it. Find the length of the whole tree.

16. Standing any where on the bank of a river, a man observed a tree exactly straight to him on the other bank that the angle journey by boat of elevation of he top of the tree of 150 metres length is 30°. The man started for the tree. But he reached at 10 metres away from the tree due to current.

(a) Show the above description by a figure.

(b) Find the width of the river.

(c) Find the distance from the starting point to the destination.

Chapter Eleven

Algebraic Ratio and Proportion

It is important for us to have a clear concept of ratio and proportion. Arithmetical ratio and proportion have been elaborately discussed in class VII. In this chapter, algebraic ratio and proportion will be discussed. We regularly use the concept of ratio and proportion in construction materials and in the production of food staff, in consumers production, in using fertilizer in land, in making the shapes and design of many things attractive and good tooking and in many areas of our daily activities. Many problems of daily lives can be solved by using ratio and proportion.

At the end of this chapter, the students will be able to :

- Explain algebraic ratio and proportion.
- Use different types of rules of transformation related to proportion.
- Describe successive proportion.
- Use ratio, proportion, successive proportion in solving real lives problem.

11·1 Ratio

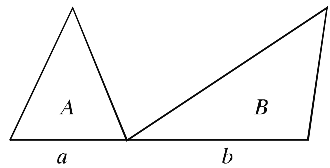
Of two quantities of same kind and unit, how many times or parts of other can be expressed by a fraction. This fraction is called the ratio of two quantities.

The ratio of two quantities p and q is written in $p : q = \frac{p}{q}$. The quantities p and q are to be of same kind and same unit. p is called antecedent and q is called subsequent of the ratio.

Some times we use ratio in approximate measure. Such as, the number of cars on the road at 8 A.M. doubles the number at 10 A.M. In this case, it is not necessary to know the exact number of cars to determine the ratio. Again, in many occasions, we say that the area of your house is three times the area of mine. Here, also it is not necessary to know the exact area of the house. We use the concept of ratio in cases of practical life.

11·2 Proportion

If four quantities are such that the ratio of first and second quantities is equal to the ratio of third and fourth quantities, those four quantities form a proportion. If a, b, c, d are four such quantities, we write $a : b = c : d$. The four quantities of proportion need not to be of same kinds. The two quantities of the ratio are to be of the same kind only.



In the above figure, let the base of two triangles be a and b respectively and their height is h unit. If the areas of the triangle be A and B square units, we can write,

$$\frac{A}{B}=\frac{\frac{1}{2}ah}{\frac{1}{2}bh}=\frac{a}{b} \quad \text{or } A:B=a:b$$

i.e. ratio of two areas is equal to ratio of two bases.

Ordered proportional

By ordered proportional of a, b, c it is meant that $a : b = b : c$.

a, b, c will be ordered proportional if and only if $b^2 = ac$. In case of ordered proportional, all the quantities are to be of same kinds. In this case, c is called third proportional of a and b and b is called mid-proportional of a and c.

Example 1. A and B traverses fixed distance in t_1 and t_2 minutes. Find the ratio of average velocity of A and B.

Solution : Let the average velocities of A and B be v_1 metre/sec and v_2 metre/sec respectively. So, in time t_1 minutes A traverses v_1t_1 metres and in t_2 minutes B traverses the distance v_2t_2 meters.

According to the problem, $v_1t_1 = v_2t_2$ $\therefore \frac{v_1}{v_2}=\frac{t_2}{t_1}$

Here, ratio of the velocities is inversely proportional to the ratio of time.

Activity: 1. Express 3.5 : 5.6 into 1 : a and b : 1   2. If $x : y = 5 : 6$, $3x : 5y =$ What ?

11·3 Transformation of Ratio

Here, the quantities of ratios are positive numbers.

(1) If $a:b=c:d$ then $b:a=d:c$ [Invertendo]

Proof : Given that, $\frac{a}{b}=\frac{c}{d}$

$\therefore ad=bc$ [multiplying both the sides by bd]

or, $\frac{ad}{ac}=\frac{bc}{ac}$ [dividing both the sides by ac where $a\neq 0, c\neq 0$]

or, $\frac{d}{c}=\frac{b}{a}$

i.e., $b : a = d : c$

(2) If $a : b = c : d$ then $a : c = b : d$ [alternendo]

Proof : Given that, $\frac{a}{b} = \frac{c}{d}$

$\therefore ad = bc$ [multiplying both the sides by bd]

or, $\frac{ad}{cd} = \frac{bc}{cd}$ [dividing both the sides by cd where $c \neq 0, d \neq 0$]

or, $\frac{a}{c} = \frac{b}{d}$

i.e., $a : c = b : d$

(3) If $a : b = c : d$ then $\frac{a+b}{b} = \frac{c+d}{d}$ [componendo]

Proof : Given that, $\frac{a}{b} = \frac{c}{d}$

$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$ [Adding 1 to both the sides]

i.e., $\frac{a+b}{b} = \frac{c+d}{d}$

(4) If $a : b = c : d$ then $\frac{a-b}{b} = \frac{c-d}{d}$ [dividendo]

Proof : $a : b = c : d$

$\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$ [subtracting 1 from both the sides]

i.e., $\frac{a-b}{b} = \frac{c-d}{d}$

(5) If $a : b = c : d$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ [componendo-dividendo]

Proof : Given that, $\frac{a}{b} = \frac{c}{d}$

By componendo, $\frac{a+b}{b} = \frac{c+d}{d}$ ....(i)

Again by dividendo, $\frac{a-b}{b} = \frac{c-d}{d}$

or, $\frac{b}{a-b} = \frac{d}{c-d}$ [by invertendo]............. (ii)

Therefore, $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$ [multiplying (i) and (ii)]

i.e., $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. [here $a \neq b$ and $c \neq d$]

(6) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$ then each of the ratio $=\frac{a+c+e+g}{b+d+f+h}$.

Proof : Let, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=k.$

$\therefore a=bk,\quad c=dk,\quad e=fk,\quad g=hk$

$\therefore \frac{a+c+e+g}{b+d+f+h}=\frac{bk+dk+fk+hk}{b+d+f+h}=\frac{k(b+d+f+h)}{b+d+f+h}=k,$

But k is equal to each of the ratio.

$\therefore \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\frac{a+c+e+g}{b+d+f+h}.$

Activity : 1. Sum of ages of mother and sister is s years. Before t years, the ratio of their ages was $r:p$. What will be the ratio of their ages after x years ?
2. The shadow of a man of height r metre, standing at p metre from a lightpost is s metre. If the height of the lightpost be h metre, what was the distance of the man from the lightpost ?

Example 2. The ratio of present ages of father and son is $7:2$, and the ratio will be $8:3$ after 5 years. What are their present ages ?

Solution : Let the present age of father be a and that of son is b. So, by the conditions of first and second of the problems, we have,

$\frac{a}{b}=\frac{7}{2}$...........(i)

$\frac{a+5}{b+5}=\frac{8}{3}$...........$(ii)$

From equation (i), we get,

$a=\frac{7b}{2}$...........(iii)

From equation (ii), we get,

$3(a+5)=8(b+5)$

or, $3a+15=8b+40$

or, $3a-8b=25$

or, $3\times\frac{7b}{2}-8b=25$ [by using (iii)]

$$\text{or, } \frac{21b-16b}{2}=25$$

$$\text{or, } 5b=50$$

$$\therefore\ b=10$$

Putting $b=10$ in equation (*iii*), we get, $a=35$

$\therefore$ The present age of father is 35 years, and that of son is 10 years.

Example 3. If $a:b=b:c$, prove that $\left(\frac{a+b}{b+c}\right)^2=\frac{a^2+b^2}{b^2+c^2}$.

Solution : Given that, $a:b=b:c$

$$\therefore\ b^2=ac$$

Now, $\left(\frac{a+b}{b+c}\right)^2=\frac{(a+b)^2}{(b+c)^2}$

$$=\frac{a^2+2ab+b^2}{b^2+2bc+c^2}$$

$$=\frac{a^2+2ab+ac}{ac+2bc+c^2}$$

$$=\frac{a(a+2b+c)}{c(a+2b+c)}=\frac{a}{c}$$

and $\frac{a^2+b^2}{b^2+c^2}=\frac{a^2+ac}{ac+c^2}$

$$=\frac{a(a+c)}{c(a+c)}$$

$$=\frac{a}{c}$$

$$\therefore\ \left(\frac{a+b}{b+c}\right)^2=\frac{a^2+b^2}{b^2+c^2}$$

Example 4. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

Solution : Let, $\frac{a}{b}=\frac{c}{d}=k$; $\therefore$ $a=bk$ and $c=dk$

Now, $\frac{a^2+b^2}{a^2-b^2}=\frac{(bk)^2+b^2}{(bk)^2-b^2}=\frac{b^2(k^2+1)}{b^2(k^2-1)}=\frac{k^2+1}{k^2-1}$

and $\frac{ac+bd}{ac-bd}=\frac{bk\cdot dk+bd}{bk\cdot dk-bd}=\frac{bd(k^2+1)}{bd(k^2-1)}=\frac{k^2+1}{k^2-1}$

$$\therefore\ \frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}.$$

Example 5. Solve : $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}=1, \quad 0<b<2a<2b.$

Solution : Given that, $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}=1$

$\therefore \sqrt{\frac{1+bx}{1-bx}}=\frac{1+ax}{1-ax}$

or, $\frac{1+bx}{1-bx}=\frac{(1+ax)^2}{(1-ax)^2}$ [squaring both the sides]

or, $\frac{1+bx}{1-bx}=\frac{1+2ax+a^2x^2}{1-2ax+a^2x^2}$

or, $\frac{1+bx+1-bx}{1+bx-1+bx}=\frac{1+2ax+a^2x^2+1-2ax+a^2x^2}{1+2ax+a^2x^2-1+2ax-a^2x^2}$ [by componendo and dividendo]

or, $\frac{2}{2bx}=\frac{2(1+a^2x^2)}{4ax}$

or, $\frac{1}{bx}=\frac{1+a^2x^2}{2ax}$

or, $2ax=bx(1+a^2x^2)$

or, $x\{2a-b(1+a^2x^2)\}=0$

$\therefore$ Either $x=0$ or $2a-b(a+a^2x^2)=0$

or, $b(1+a^2x^2)=2a$

or, $1+a^2x^2=\frac{2a}{b}$

or, $a^2x^2=\frac{2a}{b}-1$

or, $x^2=\frac{1}{a^2}\left(\frac{2a}{b}-1\right)$

$\therefore x=\pm\frac{1}{a}\sqrt{\frac{2a}{b}-1}$

$\therefore$ Required solution $x=0$, $x=\pm\frac{1}{a}\sqrt{\frac{2a}{b}-1}$.

Example 6. If $\frac{6}{x}=\frac{1}{a}+\frac{1}{b}$, show that $\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}=2, \ a\neq b.$

Solution : Given that, $\frac{6}{x}=\frac{1}{a}+\frac{1}{b}$

$\therefore 6ab=(a+b)x$ [multiplying both the sides by abx]

i.e., $x=\frac{6ab}{(a+b)}$

or, $\frac{x}{3a}=\frac{2b}{a+b}$

$\therefore \frac{x+3a}{x-3a}=\frac{2b+a+b}{2b-a-b}$ [by componendo and dividendo]

or, $\frac{x+3a}{x-3a}=\frac{a+3b}{b-a}$

Again, $\frac{x}{3b}=\frac{2a}{a+b}$

or, $\frac{x+3b}{x-3b}=\frac{2a+a+b}{2a-a-b}$ [by componendo and dividendo]

$\therefore \frac{x+3b}{x-3b}=\frac{3a+b}{a-b}$

Now, $\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}=\frac{a+3b}{b-a}+\frac{3a+b}{a-b}$

$=\frac{a+3b}{b-a}-\frac{3a+b}{b-a}=\frac{a+3b-3a-b}{b-a}=\frac{2(b-a)}{b-a}=2$.

$\therefore \frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}=2.$

Example 7. If $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=p$, prove that, $p^2-\frac{2p}{x}+1=0.$

Solution : Given that, $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=p$

$\therefore \frac{\sqrt{1+x}+\sqrt{1-x}+\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}-\sqrt{1+x}+\sqrt{1-x}}=\frac{p+1}{p-1}$ [by componendo -dividendo]

or, $\frac{2\sqrt{1+x}}{2\sqrt{1-x}}=\frac{p+1}{p-1}$ or, $\frac{\sqrt{1+x}}{\sqrt{1-x}}=\frac{p+1}{p-1}$

or, $\frac{1+x}{1-x}=\frac{(p+1)^2}{(p-1)^2}=\frac{p^2+2p+1}{p^2-2p+1}$ [squaring both sides] or, $\frac{1+x+1-x}{1+x-1+x}=\frac{p^2+2p+1+p^2-2p+1}{p^2+2p+1-p^2+2p-1}$ [by componendo -dividendo]

or, $\frac{1}{x}=\frac{p^2+1}{2p}$ or, $p^2+1=\frac{2p}{x}$

$\therefore$ $p^2-\frac{2p}{x}+1=0.$

Example 8. If $\frac{a^3+b^3}{a-b+c}=a(a+b)$, prove that a, b, c are ordered proportional.

Solution : Given that, $\frac{a^3+b^3}{a-b+c}=a(a+b)$

or, $\frac{a^3+b^3}{a-b+c}=a(a+b)$

or, $\frac{(a+b)(a^2-ab+b^2)}{a-b+c}=a(a+b)$

or, $\frac{a^2-ab+b^2}{a-b+c}=a$ [Dividing both sides by ($a+b$)]

or, $a^2-ab+b^2=a^2-ab+ac$

$\therefore$ $b^2=ac$

$\therefore$ a, b, c are ordered proportional.

Example 9. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that $c=a$ or $a+b+c+d=0$.

Solution : Given that, $\frac{a+b}{b+c}=\frac{c+d}{d+a}$

or, $\frac{a+b}{b+c}-1=\frac{c+d}{d+a}-1$ [subtracting 1 from both the sides]

or, $\frac{a+b-b-c}{b+c}=\frac{c+d-d-a}{d+a}$

or, $\frac{a-c}{b+c}=\frac{c-a}{d+a}$

or, $\frac{a-c}{b+c}+\frac{a-c}{d+a}=0$

or, $(a-c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)=0$

or, $(a-c)\dfrac{(d+a+b+c)}{(b+c)(d+a)}=0$

or, $(a-c)(d+a+b+c)=0$

$\therefore$ Either $a-c=0$ i.e., $a=c$

or, $a+b+c+d=0$.

Example 10. If $\dfrac{x}{y+z}=\dfrac{y}{z+x}=\dfrac{z}{x+y}$ and x, y, z are not mutually equal, prove that the value of each ratio is either equal -1 or equal $\dfrac{1}{2}$.

Solution : Let, $\dfrac{x}{y+z}=\dfrac{y}{z+x}=\dfrac{z}{x+y}=k$

$\therefore x=k(y+z)$................(i)

$y=k(z+x)$..................$(ii$

$z=k(x+y)$..................(iii)

Subtracting (ii) from (i), we get,

$x-y=k(y-x)$ or, $k(y-x)=-(y-x)$

$\therefore\ k=-1$

Again, adding (i), (ii) and (iii), we get,

$x+y+z=k(y+z+z+x+x+y)=2k(x+y+z)$

$\therefore\ \ \mathrm{k}=\dfrac{1(x+y+z)}{2(x+y+z)}=\dfrac{1}{2}$

$\therefore$ Therefore, the value of each of the ratio is -1 or $\dfrac{1}{2}$.

Example 11. If $ax=by=cz$, show that $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}=\dfrac{bc}{a^2}+\dfrac{ca}{b^2}+\dfrac{ab}{c^2}$.

Solution : Let, $ax=by=cz=k$

$\therefore\ x=\dfrac{k}{a},\ \ y=\dfrac{k}{b},\ \ z=\dfrac{k}{c}$

Now, $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}=\dfrac{k^2}{a^2}\times\dfrac{bc}{k^2}+\dfrac{k^2}{b^2}\times\dfrac{ca}{k^2}+\dfrac{k^2}{c^2}\times\dfrac{ab}{k^2}=\dfrac{bc}{a^2}+\dfrac{ca}{b^2}+\dfrac{ab}{c^2}$

i.e., $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}=\dfrac{bc}{a^2}+\dfrac{ca}{b^2}+\dfrac{ab}{c^2}$.

Exercise 11·1

1. If the sides of two squares be a and b metres respectively, what will be the ratio of their areas ?
2. If the area of a circle is equal to the area of a square, find the ratio of their paremetre.
3. If the ratio of two numbers is 3 : 4 and their LCM. is 180, find the two numbers.
4. The ratio of absent and present students of a day in your class is found to be 1:4. Express the number in percentage of absent students in terms of total students.
5. A thing is bought and sold at the loss of 28%. Find the ratio of buying and selling cost.
6. Sum of the ages of father and son is 70 years. 7 years ago, the ratio of their ages were 5 : 2. What will the ratio of their ages be after 5 years.
7. If $a : b = b : c$, prove that,

   (i) $\frac{a}{c}=\frac{a^2+b^2}{b^2+c^2}$ $\quad$ (ii) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3$

   (iii) $\frac{abc(a+b+c)^3}{(ab+bc+ca)^3}=1$ $\quad$ (iv) $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$

8. Solve : (i) $\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=\frac{1}{3}$ $\quad$ (ii) $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$

   (iii) $\frac{a+x-\sqrt{a^2-x^2}}{a+x+\sqrt{a^2-x^2}}=\frac{b}{x}$, $2a>b>0$ and $x\neq 0$.

   (iv) $\frac{\sqrt{x-1}+\sqrt{x-6}}{\sqrt{x-1}-\sqrt{x-6}}=5$ $\quad$ (v) $\frac{\sqrt{ax+b}+\sqrt{ax-b}}{\sqrt{ax+b}-\sqrt{ax-b}}=c$

   (vi) $81\left(\frac{1-x}{1+x}\right)^3=\frac{1+x}{1-x}$

9. If $\frac{a}{b}=\frac{c}{d}$, show that, (i) $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$

   (ii) $\frac{ac+bd}{ac-bd}=\frac{c^2+d^2}{c^2-d^2}$

10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, show that,

(i) $\dfrac{a^3+b^3}{b^3+c^3}=\dfrac{b^3+c^3}{c^3+d^3}$ (ii) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

11. If $x=\dfrac{4ab}{a+b}$, show that, $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}=2, \quad a\neq b.$

12. If $x=\dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, prove that, $x^3-3mx^2+3x-m=0$

13. If $x=\dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that, $3bx^2-4ax+3b=0.$

14. If $\dfrac{a^2+b^2}{b^2+c^2}=\dfrac{(a+b)^2}{(a+c)^2}$, prove that, a, b, c are ordered proportional.

15. If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$, prove that, $\dfrac{a}{y+z-x}=\dfrac{b}{z+x-y}=\dfrac{c}{x+y-z}$.

16. If $\dfrac{bz-cy}{a}=\dfrac{cx-az}{b}=\dfrac{ay-bx}{c}$, prove that, $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

17. If $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$ and $a+b+c\neq 0$, prove that, $a=b=c$.

18. If $\dfrac{x}{xa+yb+zc}=\dfrac{y}{ya+zb+xc}=\dfrac{z}{za+xb+yc}$ and $x+y+z\neq 0$, show that, each of the ratio is $=\dfrac{1}{a+b+c}$.

19. If $(a+b+c)p=(b+c-a)q=(c+a-b)r=(a+b-c)s$,
prove that, $\dfrac{1}{q}+\dfrac{1}{r}+\dfrac{1}{s}=\dfrac{1}{p}$.

20. If $lx=my=nz$, show that, $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}=\dfrac{mn}{l^2}+\dfrac{nl}{m^2}+\dfrac{lm}{n^2}$.

21. If $\dfrac{p}{q}=\dfrac{a^2}{b^2}$ and $\dfrac{a}{b}=\dfrac{\sqrt{a+q}}{\sqrt{a-q}}$, show that, $\dfrac{p+q}{a}=\dfrac{p-q}{q}$.

11·4. Successive Ratio

Let Roni's earning be Tk. 1000, Soni's earning be Tk. 1500 and Somi's earning be Tk. 2500. Here, Roni's earning : Soni's earning $=1000:1500=2:3$; Soni's earning : Sami's earning = 1500 : 2500 = 3 : 5. Hence, Rani's : Soni's : Sami's earning = 2 : 3 : 5.

If two ratios are of the form *a : b* and *b : c*, they can be put in the form *a : b : c*. This is called successive ratio. Any two or more than two ratios can be put in this form. It is to be noted that if two ratios are to be put in the form *a : b c*, antecedent of the first ratio and subsequent of the second ratio are to be equal. Such as, if two ratios 2 : 3 and 4 : 3 are to be put in the form *a : b* : *c* the subsequent quantity of first ratio is to be made equal to antecedent quantity of the second ratio. That is those quantities are to be made equal to their LCM.

Here, $2:3=\frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12}=8:12$ Again, $4:3=\frac{4}{3}=\frac{4\times 3}{3\times 3}=\frac{12}{9}=12:9$

Therefore, if the ratios 2 : 3 and 4 : 3 are put in the form, *a : b : c* will be 8 : 12 : 9.

It is to be noted that if the earning of Sami in the above example is 1125, the ratio of their earnings will be 8 : 12 : 9.

Example 12. If *a, b, c* are quantities of same kind and *a : b* = 3 : 4, *b : c* = 6 : 7, what will be *a : b : c* ?

Solution : $\frac{a}{b}=\frac{3}{4}=\frac{3\times 3}{4\times 3}=\frac{9}{12}$ and $\frac{b}{c}=\frac{6}{7}=\frac{6\times 2}{7\times 2}=\frac{12}{14}$ [LCM. of 4 and 6 is 12]

$\therefore$ *a : b : c* = 9 : 12 : 14.

Example 13. The ratio of angles of a triangle is 3 : 4 : 5. Express the angles in degree.

Solution : Sum of three angles = 180°

Let the angles, according to given ratio, be $3x$, $4x$ and $5x$.

According to the problem, $3x+4x+5x=180^\circ$ or, $12x=180^\circ$ or, $x=15^\circ$

Therefore, the angles are $3x=3\times 15^\circ=45^\circ$

$4x=4\times 15^\circ=60^\circ$

and $5x=5\times 15^\circ=75^\circ$

Example 14. If the sides of a square increase by 10%, how much will the area be increased in percentage ?

Solution : Let each side of the square be a metre

$\therefore$ Area of the square be a^2 square metre

If the side increases by 10%, each side will be $(a + 10\% \text{ of } a)$ metre or $1{\cdot}10a$ metre.

In this case, the area of the square will be $(1 \cdot 10a)^2$ square metre or, $1 \cdot 21a^2$ square metre.

Area increases by $(1 \cdot 21a^2 - a^2) = 0.21a^2$ square metre

$\therefore$ The percentage of increment of the area will be $\dfrac{0 \cdot 21a^2}{a^2} \times 100\% = 21\%$

Activity :

1. There are 35 male and 25 female students in your class. The ratios of rice and pulse are 3 : 1 and 5 : 2 given by each of the male and female students for taking khisuri in a picnic. Find the ratio of total rice and total pulse.

11·5 Proportional Division

Division of a quantity into fixed ratio is called proportional division. If S is to be divided into $a : b : c : d$, dividing S by $(a + b + c + d)$ the parts a, b, c and d are to be taken.

Therefore,

1st part $= \dfrac{a}{a+b+c+d}$ of $S = \dfrac{Sa}{a+b+c+d}$

2nd part $= \dfrac{b}{a+b+c+d}$ of $S = \dfrac{Sb}{a+b+c+d}$

3rd part $= \dfrac{c}{a+b+c+d}$ of $S = \dfrac{Sc}{a+b+c+d}$

4th part $= \dfrac{d}{a+b+c+d}$ of $S = \dfrac{Sd}{a+b+c+d}$

In this way, any quantity may be divided into any fixed ratio.

Example 15. Divide Tk. 5100 among 3 persons in such a way that 1st person's part : 2nd person's part : 3rd person's part are $=\frac{1}{2}:\frac{1}{3}:\frac{1}{9}$.

Solution : Here, $\frac{1}{2}:\frac{1}{3}:\frac{1}{9}=\left(\frac{1}{2}\times18\right):\left(\frac{1}{3}\times18\right):\left(\frac{1}{9}\times18\right)$ [L.C.M. of 2, 3, 9 is 18]

$=9:6:2$

Sum of the quantities of ratio $=9+6+2=17$.

1st person's part $=\text{Tk. }5100\times\frac{9}{17}=$ Tk. 2700

2nd person's part $=\text{Tk. }5100\times\frac{6}{17}=$ Tk. 1800

3rd person's part $=\text{Tk. }5100\times\frac{2}{17}=$ Tk. 600

Therefore, three persons will have Tk. 2700, Tk. 1800 and Tk. 600 respectively.

Exercise 11·2

1. If a,b,c are ordered proportional, which one is correct of the followings ?

   (a) $a^2=bc$ (b) $b^2=ac$ (c) $ab=bc$ (d) $a=b=c$

2. The ratio of ages of Asif and Akib is 5 : 3 ; if Asif is of 20 years old, how many years later the ratio of their ages will be 7 : 5. ?

   (a) 5 years (b) 6 years (c) 8 years (d) 10 years

3. Note the following information :

   (i) All the four quantities need not to be of same kind in proportion.

   (ii) The ratio of areas of two triangles is equal to the ratio of areas of their bases.

   (iii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, value of each ratio will be $\frac{a+c+e+g}{b+d+f+h}$.

On the basis of above information, which one of the followings is correct?

(a) (i) and (ii) (b) (ii) and (iii)

(c) (i) and (iii) (d) (i), (ii) and (iii)

The ratio of angles of ΔABC is 2 : 3 : 5 and the ratio of angles of the quadrilateral $ABCD$ is 3 : 4 : 5 : 6. Based on this information, answer questions number 4 and 5.

4. If the sides of a square double, how much will the area of a square be increased?

(a) 2 times (b) 4 times

(c) 8 times (d) 6 times

5. If $x : y - 7 : 5,\ y : z = 5 : 7$, $z : x$ = how much ?

6. The estimated cost for the construction of a wooden bridge is Tk. 90,000. But Tk. 21,000 has been spent more. What is the percentage of the excess cost ?

7. The ratio of rice and husk in paddy to 7 : 3. What is the percentage of rice in it ?

8. The weight ot 1 cubic cm. wood is 7 decigram. What is the percentage of the weight of wood to the equivalent volume of water ?

9. Distribute Tk. 300 among *a, b, c, d* in such a way that the raios are *a*'s part : *b*'s part = 2 : 3, *b*'s part : *c*'s part = 1 : 2 and *c*'s part : *d*'s part 3 : 2.

10. Three fishermen have caught 690 pieces of fishes. The ratios of their parts are $\frac{2}{3}, \frac{4}{5}$ and $\frac{5}{6}$. How many fishes each of them will get. ?

11. The parametre of a triangle is 45 cm. The ratio of the lengths of the sides is 3 : 5 : 7. Find the length of each sides.

12. Distribute Tk. 1011 in the ratio $\frac{3}{4} : \frac{4}{5} : \frac{6}{7}$.

13. If the ratio of two numbers is 5 : 7 and their H.C.F. is 4, what is L.C.M. of the numbers.

14. In a cricket game, the total runs scored by Sakib, Mushfique and Mashrafi were 171. The ratio of runs scored by Sakib and Mushfiques, and Mushfique and Mashrafi was 3 : 2. What were the runs scored by them individually.
15. In a office, there were 2 officers, 7 clarks and 3 bearers. If a bearer gets Tk. 1, a clerk gets Tk. 2 and an officer gets Tk. 4. Their total salary is Tk. 15,000. What is their individual salary ?
16. In selecting the leader of a society, Mr. Denal won in ratio of 4 : 3 votes of the two contestants. If total numbers of members were 581 and 91 members did not cast their votes, what was the difference of votes by which opposite of Mr. Denal had been defeated ?
17. If the sides of a square are increased by 20%, what is percentage of increment of the area of the square ?
18. If the length of a rectangle is increased by 10% and the breadth is decreased by 10%, what is the percentage of increase or decrease of the area of the rectangle?
19. In a field, the ratio of production is 4 : 7 before and after irrigation. In that field, the production of paddy in a land previously was 304 quintal. What would be the production of paddy after irrigation ?
20. If the ratio of paddy and rice produced from paddy is 3 : 2 and the ratio of wheat and suzi produced from wheat is 4 : 3, find the ratio of rice and suzi produced from equal quantity of rice and wheat.
21. The are of a land is 432 square metre. If the ratios of lengths and breadths of that land and that of another land be 3 : 4 and 2 : 5 respectively, that what is the area of another land ?
22. Zami and Simi take loans of different amounts at the rate of 10% simple profit on the same day from same Bank. The amount on capital and profit which Zimi refunds after two years, the same amount Simi refunds after three years on capital and profit. Find the ratio of their loan.
23. The ratio of sides of a triangle is 5 : 12 : 13 and parametre is 30 cm.

    (a) Draw the triangle and write what type of triangle in respect of angles.

(b) Determine the area of a square drawn with the diagonal of a rectangle taking greater side as length and smaller side as breadth as the sides of a square.

(c) If the length is increased by 10% and breadth is increased by 20%, what will be percentage of increase of the area ?

24. The ratio of present and absent of students of a day in a class is 1 : 4.

(a) Express the percentage of absent students against total students.

(b) The ratio of present and absent students would be 1 : 9 if 10 more students were present. What was the total number of students ?

(c) Of the total number of students, the number of female students is less than male students by 20. Find the ratio of male and female students.

Chapter Twelve

Simple Simultaneous Equations with Two Variables

For solving the mathematical problems, the most important topic of Algebra is equation. In classes V and VI, we have got the idea of simple equation and have known how to solve the simple equation with one variable. In class VII, we have solved the simple simultaneous equations by the methods of substitution and elimination and by graphs. We have also learnt how to form and solve simple simultaneous equations related to real life, problems. In this chapter, the idea of simple simultaneous equations have been expanded and new methods of solution have been discussed. Besides, in this chapter, solution by graphs and formation of simultaneous equations related to real life problems and their solutions have been discussed in detail.

At the end of the chapter, the students will be able to –

- Verify the consistency of simple simultaneous equations with two variables.
- Verify the mutual dependence of two simple simultaneous equations with two variables
- Explain the method of cross-multiplication
- Form and solve simultaneous equations related to real life mathematical problems
- Solve the simultaneous equations with two variables by graphs.

12·1 Simple simultaneous equations.

Simple simultaneous equations means two simple equations with two variables when they are presented together and the two variables are of same characteristics. Such two equations together are also called system of simple equations. In class VII, we have solved such system of equations and learnt to form and solve simultaneous equations related to real life problems. In this chapter, these have been discussed in more details.

First, we consider the equation $2x + y = 12$. This is a simple equation with two variables.

In the equation, can we get such values of x and y on the left hand side for which the sum of twice the first with the second will be equal to 12 of the right hand side ; that is, the equation will be satisfied by those two values ?

Now, we fill in the following chart from the equation $2x + y = 12$:

Value of x	Value of y	Value of L.H.S $(2x+y)$	R.H.S
-2	16	$-4+16=12$	12
0	12	$0+12=12$	12
3	6	$6+6=12$	12
5	2	$10+2=12$	12
....	.....	..... $=12$	12

The equation has infinite number of solutions. Among those, four solutions are $(-2,16)$, $(0,12)$, $(3,6)$ and $(5,2)$.

Again, we fill in the following chart from another equation $x-y=3$:

Value of x	Values of y	Value of L.H.S $(x-y)$	R.H.S
-2	-5	$-2+5=3$	3
0	-3	$0+3=3$	3
3	0	$3-0=3$	3
5	2	$5-2=3$	3
....	.....	..... $=3$	3

The equation has infinite number of solutions. Among those, four solutions are $(-2,-5)$, $(0,-3)$, $(3,0)$ and $(5,2)$.

If the two equations discussed above are considered together a system, both the equations will be satisfied simultaneously only by (5, 2). Both the equations will not be satisfied simultaneously by any other values.

Therefore, the solution of the system of equations $2x+y=12$ and $x-y=3$ is $(x,y)=(5,2)$

> **Activity :** Write down five solutions for each of the two equations $x-2y+1=0$ and $2x+y-3=0$ so that among the solutions, the common solutions also exists.

12·2 Conformability for the solution of simple simultaneous equations with two variables.

(a) As discussed earlier, the system of equations $\left.\begin{aligned}2x+y&=12\\x-y&=3\end{aligned}\right\}$ have unique (only one) solution. Such system of equations are called consistent. Comparing the coefficient of x and y (taking the ratio of the coefficients) of the two equations, we get, $\frac{2}{1}\neq\frac{1}{-1}$; any equation of the system of equations cannot be expressed in terms of the other. That is why, such system of equations are called mutually independent. In the case of consistent and mutually independent system of equations, the ratios are not equal. In this case, the constant terms need not to be compared.

(b) Now we shall consider the system of equations $\left.\begin{array}{l}2x-y\ \ =6\\4x-2y=12\end{array}\right\}$. Will this two equations be solved ?

Here, if both sides of first equation are multiplied by 2, we shall get the second equation. Again, if both sides of second equation are divided by 2, we shall get the first equation. That is, the two equations are mutually dependent.

We know, first equation has infinite number of solutions. So, 2nd equation has also the same infinite number of solutions. Such system of equations are called consistent and mutually dependent. Such system of equations have infinite number of solutions.

Here, comparing the coefficients of x, y and the constant terms of the two equations, we get, $\frac{2}{4}=\frac{-1}{-2}=\frac{6}{12}\left(=\frac{1}{2}\right)$.

That is, in the case of the system of such simultaneous equations, the ratios become equal.

(c) Now, we shall try to solve the system of equations $\left.\begin{array}{l}2x+y\ \ =12\\4x+2y=5\end{array}\right\}$.

Here, multiplying both sides of first equation by 2, we get, $4x+2y=24$

second equation is $4x+2y=\ 5$

subtracting, $0=19$, which is impossible.

So, we can say, such system of equations cannot be solved. Such system of equations are inconsistent and mutually independent. Such system of equations have no solution.

Here, comparing the coefficients of x, y and constant terms from the two equations, we get, $\frac{2}{4}=\frac{1}{2}\neq\frac{12}{5}$. That is, in case of the system of inconsistent and mutually independent equations ratios of the coefficients of the variables are not equal to the ratio of the constant terms. Generally, conditions for comformability of two simple simultaneous equations, such as, $\left.\begin{array}{l}a_1x+b_1y\ \ =c_1\\a_2x+b_2y=c_2\end{array}\right\}$ are given in the chart below :

	system of equations	**Comparison of coeff.and const. terms**	**consistent/ incon sistent**	**mutually dependent/ independent**	**has solution (how many) / no.**
(i)	$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$	$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$	consistent	independent	Yes (only one)
(ii)	$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$	consistent	dependent	Yes (infinite numbers)
(iii)	$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$	inconsistent	independent	No

Now, if there is no constant terms in both the equations of a system of equations ; i.e., $c_1 = c_2 = 0$, if with reference to the above discussion from (i), if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ the system of equations are always consistent and independent of each other. In that case, there will be only one (unique) solution.

From (*ii*) and (*iii*) if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, the system of equations are consistent and dependent of each other. In that case, there will be infinite number of solutions.

Example : Explain whether the following system of equations are consistent / inconsistent, dependent/ independent of each other and indicate the number of solutions in each case.

(a) $x + 3y = 1$
$2x + 6y = 2$

(b) $2x - 5y = 3$
$x + 3y = 1$

(c) $3x - 5y = 7$
$6x - 10y = 15$

Solution :

(a) Given system of equations are : $\left.\begin{aligned} x + 3y &= 1 \\ 2x + 6y &= 2 \end{aligned}\right\}$

Ratio of the coefficients of x is $\frac{1}{2}$

Ratio of the coefficients of y is $\frac{3}{6}$ or $\frac{1}{2}$

Ratio of constant terms is $\frac{1}{2}$

$\therefore \quad \frac{1}{2} = \frac{3}{6} = \frac{1}{2}$

Therefore, the system of equations are consistent and mutually dependent. The system of equations have infinite number of solutions.

(b) Given system of equations are : $\left.\begin{aligned} 2x - 5y &= 3 \\ x + 3y &= 1 \end{aligned}\right\}$

Ratio of the coefficients of x is $\frac{2}{1}$

Ratio of the coefficients of y is $\frac{-5}{3}$

$\therefore$ we have, $\frac{2}{1} \neq \frac{-5}{3}$

Therefore, the system of equations are consistent and mutually independent. The system of equations have only one (unique) solution.

(c) Given system of equations are : $3x - 5y = 7$

$$6x - 10y = 15$$

Ratio of the coefficients of x is $\frac{3}{6}$ or $\frac{1}{2}$

Ratio of the coefficients of y is $\frac{-5}{-10}$ or $\frac{1}{2}$

ratio of the constant terms is $\frac{7}{15}$

$\therefore$ we get, $\frac{3}{6} = \frac{-5}{-10} \neq \frac{7}{15}$

Therefore, the system of equations are inconsistent and mutually independent. The system of equations have no solution.

Activity : Verify whether the system of equations $x - 2y + 1 = 0, 2x + y - 3 = 0$ are consistent and dependent and indicate how many solutions the system of equations may have.

Exercise 12·1

Mention with arguments, whether the following simple simultaneous equations are consistent/inconsistent, mutually dependen t/independent and indicate the number of solutions :

1. $x - y = 4$, $x + y = 10$
2. $2x + y = 3$, $4x + 2y = 6$
3. $x - y - 4 = 0$, $3x - 3y - 10 = 0$
4. $3x + 2y = 0$, $6x + 4y = 0$
5. $3x + 2y = 0$, $9x - 6y = 0$
6. $5x - 2y - 16 = 0$, $3x - \frac{6}{5}y = 2$
7. $-\frac{1}{2}x + y = -1$, $x - 2y = 2$
8. $-\frac{1}{2}x - y = 0$, $x - 2y = 0$
9. $-\frac{1}{2}x + y = -1$, $x + y = 5$
10. $ax - cy = 0$, $cx - ay = c^2 - a^2$.

12·3 Solution of simple sumultaneous equations

We shall discuss the solutions of only the consistent and independent simple simultaneous equations. Such system of equation has only one (unique) solution.

Here, four methods of solutions are discussed :

(1) Method of substitution, (2) Method of elimination (3) Method of cross-multiplication (4) Graphical method.

In class VII, we have known how to solve by the methods of substitution and elimination. Here, examples of one for each of these two methods are given.

Example 1. Solve by the method of substitution :

$$2x + y = 8$$
$$3x - 2y = 5$$

Solution : Given equations are :

$$2x + y = 8 \text{.............}(1)$$
$$3x - 2y = 5 \text{...........}(2)$$

From equation (1), $y = 8 - 2x$.........(3)

Putting the value of y form equation (3) in equation (2), we get

$3x - 2(8 - 2x) = 5$

or, $3x - 16 + 4x = 5$

or, $3x + 4x = 5 + 16$

or, $7x = 21$

or, $x = 3$

Putting the value of x in equation (3)

$y = 8 - 2 \times 3$

$= 8 - 6$

$= 2$

$\therefore$ Solution $(x, y) = (3, 2)$

Solution by the method of substitution :

Conveniently, from any of the two equations, value of one variable is expressed in terms of the other variable and putting the obtained value in the other equation, we shall get an equation with one variable. Solving this equation, value of the variable can be found. This value can be put in any of the equations. But, if it is put in the equation in which one variable has been expressed in terms of the other variable, the solution will be easier. From this equation, value of the other variable will be found.

Example 2. Solve by the method of elimination : $2x + y = 8$
$3x - 2y = 5$

[N.B. : To show the difference between the methods of substitution and elimination, same equations of example 1 have been taken in this example 2]

Solution : Given equations are $2x + y = 8$.............(1)
$3x - 2y = 5$...........(2)

Multiplying both sides of equation (1) by 2, $4x + 2y = 16$.............(3)

equation (2) is $3x - 2y = 5$...........(2)

Adding (3) and (2), $7x = 21$ or $x = 3$.

Putting the value of x in equation (1), we get

$2 \times 3 + y = 8$

or, $y = 8 - 6$

or, $y = 2$

$\therefore$ Solution $(x, y) = (3, 2)$

Solution by the method of elimination :

Conveniently, one equation or both equations are multiplied by such a number so that after multiplication, absolute value of the coefficients of the same variable become equal. Then as per need, if the equations are added or subtracted, the variable with equal coefficient will be eliminated. Then, solving the obtained equation, the value of the existing variable will be found. If that value is put conveniently in any of the given equations, value of the other variable will be found.

(3) Method of cross-multiplication :

We consider the following two equations :

$$a_1x + b_1y + c_1 = 0..........(1)$$

$$a_2x + b_2y + c_2 = 0..........(2)$$

Multiplying equation (1) by b_2 and equation (2) by b_1, we get,

$$a_1b_2x + b_1b_2y + b_2c_1 = 0.........(3)$$

$$a_2b_1x + b_1b_2y + b_1c_2 = 0.........(4)$$

Subtracting equation (4) from equation (3), we get

$$(a_1b_2 - a_2b_1)x + b_2c_1 - b_1c_2 = 0$$

$$\text{or, } (a_1b_2 - a_2b_1)x = b_1c_2 - b_2c_1$$

$$\text{or, } \frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}..........(5)$$

Again, multiplying equation (1) by a_2 and equation (2) by a_1, we get,

$$a_1a_2x + a_2b_1y + c_1a_2 = 0.........(6)$$

$$a_1a_2x + a_1b_2y + c_2a_1 = 0.........(7)$$

Subtracting equation (7) from equation (6), we get

$$(a_2b_1 - a_1b_2)y + c_1a_2 - c_2a_1 = 0$$

$$\text{or, } -(a_1b_2 - a_2b_1)y = -(c_1a_2 - c_2a_1)$$

$$\text{or, } \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}..........(8)$$

From (5) and (8) we get,

$$\boxed{\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}}$$

From such relation between x and y, the technique of finding their values is called the method of cross-multiplication.

From the above relation between x and y, we get,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}\ , \quad \text{or} \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Again, $\dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$, or $y = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

$\therefore$ The solution of the given equations : $(x, y) = \left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

We observe :

Equations	Relation between x and y	Illustration
$a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$	$\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$	$x \quad y \quad 1$ $a_1 \mid b_1 \quad c_1 \quad a_1 \quad b_1$ $a_2 \mid b_2 \quad c_2 \quad a_2 \quad b_2$

[N.B. : The method of cross-multiplication can also be applied by keeping the constant terms of both equations on the right hand side. In that case, changes of sign will occur ; but the solution will remain same.]

Activity : If the system of equations

$\left.\begin{array}{l} 4x - y - 7 = 0 \\ 3x + y \quad = 0 \end{array}\right\}$ are expressed as the system of equations $\left.\begin{array}{l} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{array}\right\}$,

find the values of $a_1, b_1, c_1, a_2, b_2, c_2$.

Example 3. Solve by the method of cross-multiplication : $6x - y = 1$
$3x + 2y = 13$

Solution : Making the right hand side of the equations 0 (zero) by transposition, we get,

$6x - y - 1 = 0$

$3x + 2y - 13 = 0$

comparing the equations with

$\left.\begin{array}{l} a_1x + b_1y_2 + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{array}\right\}$ respectively,

we get, $a_1 = 6, b_1 = -1, c_1 = -1$
$a_2 = 3, b_2 = 2, c_2 = -13$

By the method of cross-multiplication, we get,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Illustration :

$x \quad y \quad 1$

$a_1 \mid b_1 \quad c_1 \quad a_1 \quad b_1$

$a_2 \mid b_2 \quad c_2 \quad a_2 \quad b_2$

or $\dfrac{x}{(-1)\times(-13)-2\times(-1)}=\dfrac{y}{(-1)\times 3-(-13)\times 6}=\dfrac{1}{6\times 2-3\times(-1)}$

$$\begin{array}{c|cccc} & x & y & 1 & \\ 6 & -1 & -1 & 6 & -1 \\ 3 & 2 & -13 & 3 & 2 \end{array}$$

or $\dfrac{x}{13+2}=\dfrac{y}{-3+78}=\dfrac{1}{12+3}$

or $\dfrac{x}{15}=\dfrac{y}{75}=\dfrac{1}{15}$

$\therefore$ $\dfrac{x}{15}=\dfrac{1}{15}$, or $x=\dfrac{15}{15}=1$

Again, $\dfrac{y}{75}=\dfrac{1}{15}$, or $y=\dfrac{75}{15}=5$

$\therefore$ Solution $(x,y)=(1,5)$

Example 4. Solve by the method of cross-multiplication : $3x-4y=0$
$2x-3y=-1$

Solution : Given equations are :

$$\left.\begin{array}{l}3x-4y=0\\2x-3y=-1\end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l}3x-4y+0=0\\2x-3y+1=0\end{array}\right\}$$

By the method of cross-multiplication, we get,

$$\frac{x}{-4\times 1-(-3)\times 0}=\frac{y}{0\times 2-1\times 3}=\frac{1}{3\times(-3)-2\times(-4)}$$

$$\begin{array}{c|cccc} & x & y & 1 & \\ 3 & -4 & 0 & 3 & -4 \\ 2 & -3 & 1 & 2 & -3 \end{array}$$

or $\dfrac{x}{-4+0}=\dfrac{y}{0-3}=\dfrac{1}{-9+8}$

or $\dfrac{x}{-4}=\dfrac{y}{-3}=\dfrac{1}{-1}$

or $\dfrac{x}{4}=\dfrac{y}{3}=\dfrac{1}{1}$

$\therefore$ $\dfrac{x}{4}=\dfrac{1}{1}$, or $x=4$

Again, $\dfrac{y}{3}=\dfrac{1}{1}$, or $y=3$

$\therefore$ Solution $(x,y)=(4,3)$.

Example 5. Solve by the method of cross-multiplication : $\dfrac{x}{2}+\dfrac{y}{3}=8$
$\dfrac{5x}{4}-3y=-3$

Solution : Arranging the given equations in the form $ax+by+c=0$, we get,

$\frac{x}{2}+\frac{y}{3}=8$

or $\frac{3x+2y}{6}=8$

or $3x+2y-48=0$

Again, $\frac{5x}{4}-3y=-3$

or $\frac{5x-12y}{4}=-3$

or $5x-12y+12=0$

$\therefore$ the given equations are : $3x+2y-48=0$
$5x-12y+12=0$

By the method of cross-multiplication, we get,

$$\frac{x}{2\times12-(-12)\times(-48)}=\frac{y}{(-48)\times5-12\times3}=\frac{1}{3\times(-12)-5\times2}$$

	x		y		1	
3	2	-48		3		2
5	-12	12		5		-12

or $\frac{x}{24-576}=\frac{y}{-240-36}=\frac{1}{-36-10}$

or $\frac{x}{-552}=\frac{y}{-276}=\frac{1}{-46}$

or $\frac{x}{552}=\frac{y}{276}=\frac{1}{46}$

$\therefore \frac{x}{552}=\frac{1}{46}$ or, $x=\frac{552}{46}=12$

Again, $\frac{y}{276}=\frac{1}{46}$, or $y=\frac{276}{46}=6$

$\therefore$ Solution $(x, y)=(12,6)$

Verification of the correctness of the solution :

Putting the values of x and y in given equations, we get,

In 1st equation, L.H.S $=\frac{x}{2}+\frac{y}{3}=\frac{12}{2}+\frac{6}{3}=6+2$
$=8=$ R.H.S

In 2nd equation, L.H.S $=\frac{5x}{4}-3y=\frac{5\times12}{4}-3\times6$
$=15-18=-3=$ R.H.S

$\therefore$ the solution is correct.

Example 6. Solve by the method of cross-multiplication : $ax-by=ab=bx-ay$.

Solution : Given equations are

$$\left.\begin{array}{l}ax-by=ab\\ bx-ay=ab\end{array}\right\} \text{ or, } \left.\begin{array}{l}ax-by-ab=0\\ bx-ay-ab=0\end{array}\right\}$$

By the method of cross-multiplication, we get,

$$\therefore \frac{x}{(-b)\times(-ab)-(-a)(-ab)}=\frac{y}{(-ab)\times b-(-ab)\times a}=\frac{1}{a\times(-a)-b\times(-b)}$$

	x	y	1	
a	−b	−ab	a	−b
b	−a	−ab	b	−a

or $\frac{x}{ab^2-a^2b}=\frac{y}{-ab^2+a^2b}=\frac{1}{-a^2+b^2}$

or $\frac{x}{-ab(a-b)}=\frac{y}{ab(a-b)}=\frac{1}{-(a+b)(a-b)}$

or $\frac{x}{ab(a-b)}=\frac{y}{-ab(a-b)}=\frac{1}{(a+b)(a-b)}$

$\therefore \frac{x}{ab(a-b)}=\frac{1}{(a+b)(a-b)}$, or $x=\frac{ab(a-b)}{(a+b)(a-b)}=\frac{ab}{a+b}$

Again, $\frac{y}{-ab(a-b)}=\frac{1}{(a+b)(a-b)}$, or $y=\frac{-ab(a-b)}{(a+b)(a-b)}=\frac{-ab}{a+b}$

$\therefore \ (x,y)=\left(\frac{ab}{a+b},\frac{-ab}{a+b}\right)$

Exercise 12·2

Solve by the method of substitution (1 -3) :

1. $7x-3y=31$, $9x-5y=41$
2. $\frac{x}{2}+\frac{y}{3}=1$, $\frac{x}{3}+\frac{y}{2}=1$
3. $\frac{x}{a}+\frac{y}{b}=2$, $ax+by=a^2+b^2$

Solve by the method of elimination (4 -6) :

4. $7x-3y=31$, $9x-5y=41$
5. $7x-8y=-9$, $5x-4y=-3$
6. $ax+by=c$, $a^2x+b^2y=c^2$

Solve by the method of cross-multiplication (7 -15) :

7. $2x+3y+5=0$, $4x+7y+6=0$
8. $3x-5y+9=0$, $5x-3y-1=0$
9. $x+2y=7$, $2x-3y=0$
10. $4x+3y=-12$, $2x=5$
11. $-7x+8y=9$, $5x-4y=-3$
12. $3x-y-7=0=2x+y-3$
13. $ax+by=a^2+b^2$, $2bx-ay=ab$
14. $y(3+x)=x(6+y)$, $3(3+x)=5(y-1)$
15. $(x+7)(y-3)+7=(y+3)(x-1)+5$

12·4 Solution by graphical method

In a simple equation with two variables, the relation of existing variables x and y can be expressed by picture. This picture is called the graphs of that relation. In the graph of such equation, there exist infinite number of points. Plotting a few such points, if they are joined with each other, we shall get the graph.

Each of a simple simultaneous equations has infinite number of solutions. Graph of each equation is a straight line. Coordinates of each point of the straght line satisfies the equation. To indicate a graph, two or more than two points are necessary.

Now we shall try to solve graphically the following system of equations :

$2x + y = 3$..........(1)

$4x + 2y = 6$........(2)

From equation (1), we get, $y = 3 - 2x$.

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table :

x	-1	0	3
y	5	3	-3

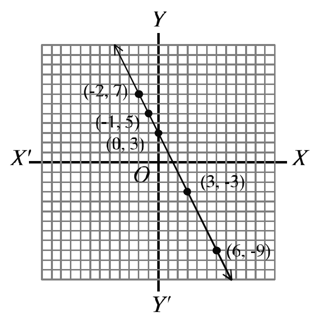
$\therefore$ three points on the graph of the equation are : $(-1, 5), (0, 3)$ and $(3, -3)$ |

Again, from equation (2), we get, $2y = 6 - 4x$ or, $y = \frac{6 - 4x}{2}$

Taking some values of x in the equations, we find the the corresponding values of y and make the adjoining table :

x	-2	0	6
y	7	3	-9

$\therefore$ three points on the graph of the equation are : $(-2, 7), (0, 3)$ and $(6, -9)$



In a graph paper let XOX' and YOY' be respectively the x axis and y axis and O is the origin.

We take each side of smallest squares of the graph paper as unit along with both axes. Now, we plot the points $(-1, 5), (0, 3)$ and $(3, -3)$ obtained from equation (1) and join them each other. The graph is a straight line.

Again, we plot the points $(-2, 7), (0, 3)$ and $(6, -9)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line. But we observe that the two straight lines coincide and they have turned into the one straight line. Again, if both sides of equation (2) are divided by 2, we get he equation (1). That is why the graphs of the two equations coincide.

Here, the system of equations, $\left.\begin{array}{l} 2x + y = 3..........(1) \\ 4x + 2y = 6........(2) \end{array}\right\}$ are consistent and mutually dependent. Such system of equations have infinite number of solutions and its graph is a straight line.

Now, we shall try to solve the system of equations : $2x - y = 4$..........(1)
$4x - 2y = 12$......(2)

From equation (1), we get, $y = 2x - 4$.

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table :

x	-1	0	4
y	-6	-4	4

$\therefore$ three points on the graph of the equation are : $(-1, -6), (0, -4), (4, 4)$.

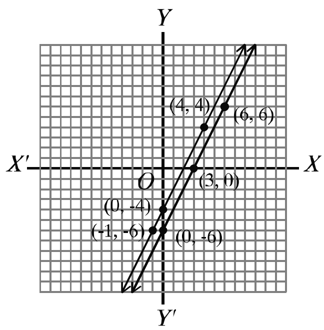
Again, from equation (2), we get,
$4x - 2y = 12$, or $2x - y = 6$
or $y = 2x - 6$

Taking some values of x in the equation, we find the the corresponding values of y and make the adjoining table :

x	0	3	6
y	-6	0	6

$\therefore$ three points on the graph of the equation are : $(0, -6), (3, 0), (6, 6)$

In graph paper let XOX' and YOY' be respectively x axis and y axis and O is the origin.



Taking each side of smallest squares in the graph paper as unit, we plot the points $(-1, -6), (0, -4)$ and $(4, 4)$ obtained from equation (1) and join them each other. The graph is a straight line.

Again, we plot the points $(0, -6), (3, 0), (6, 6)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.

We observe in the graph, though each of the given equations has separately infinite number of solutions, they have no common solution as system of simultaneous equations. Further, we observe that the graphs of the two equations are straight lines parallel to each other. That is, the lines will never intersect each other. Therefore, there will be no common point of the lines. In this case we say, such system of equations have no solution. We know, such system of equations are inconsistent and independent of each other.

Now, we shall solve the system of two consistent and independent equations by graphs. Graphs of two such equations with two variables intersect at a point. Both the equations will be satisfied by the coordinates of that point. The very coordinates of the point of intersection will be the solution of the two equations.

Example 7. Solve and show the solution in graph : $2x + y = 8$
$3x - 2y = 5$

Solution : Given two equations are : $2x + y - 8 = 0$...........(1)
$3x - 2y - 5 = 0$............(2)

By the method of cross-multiplication, we get,

$$\frac{x}{1\times(-5)-(-2)\times(-8)}=\frac{y}{(-8)\times3-(-5)\times2}=\frac{1}{2(-2)-3\times1}$$

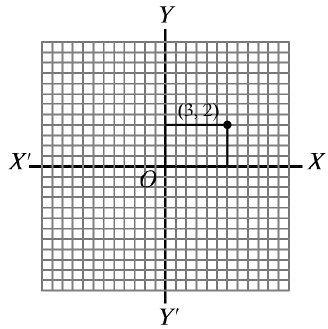
or $\frac{x}{-5-16}=\frac{y}{-24+10}=\frac{1}{-4-3}$

or $\frac{x}{-21}=\frac{y}{-14}=\frac{1}{-7}$

or $\frac{x}{21}=\frac{y}{14}=\frac{1}{7}$

$\therefore$ $\frac{x}{21}=\frac{1}{7}$, or $x=\frac{21}{7}=3$

Again, $\frac{y}{14}=\frac{1}{7}$, or $y=\frac{14}{7}=2$



$\therefore$ solution is $(x,y)=(3,2)$

Let XOX' and YOY' be x axis and y axis respectively and O, the origin.

Taking each two sides of the smallest squares along with both axes of the graph paper as one unit, we plot the point (3, 2).

Example 8. Solve with the help of graphs :

$$3x-y=3$$
$$5x+y=21$$

Solution : : Given equations are : $3x-y=3$..........(1)

$5x+y=21$........(2)

x	-1	0	3
y	-6	-3	6

From equation (1), we get, $3x-y=3$, or $y=3x-3$

Taking some values of x in equation (1), we get corresponding values of y and make the table beside :

$\therefore$ three points on the graph of the equation are : $(-1,-6),(0,-3),(3,6)$

Again, from equation (2), we get, $5x+y=21$, or $y=21-5x$

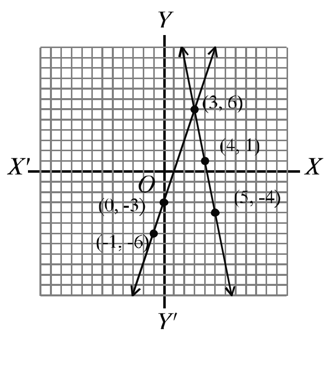
Taking some values of x in equation (2), we find the corresponding values of y and make the adjoining table :

x	3	4	5
y	6	1	-4

$\therefore$ three points on the graph of the equation are : $(3,6),(4,1),(5,-4)$.

Let XOX' and YOY' be respectively x axis and y axis and O be the origin.

We take each side of the smallest square of the graph paper as unit.



Now, we plot the points $(-1,-6),(0,-3),(3,6)$ obtained from equation (1) and join them successively. The graph is a straight line. Similarly, we plot the points $(3,6),(4,1),(5,-4)$ obtained from equation (2) and join them successively. In this case also the graph is a straight line. Let the two straight lines intersect each other at P. It is seen from the picture that the coordinates of P are $(3,6)$.

$\therefore$ solution is $(x, y) = (3,6)$

Example 9. Solve by graphical method : $2x+5y=-14$
$4x-5y=17$

Solution : Given equations are : $2x+5y=-14$...........(1)
$4x-5y=17$........(2)

From equation (1), we get, $5y=-14-2x$, or $y=\frac{-2x-14}{5}$

Taking some convenient values of x in the equation, we find the corresponding values of y and make the adjoining table :

x	3	$\frac{1}{2}$	-2
y	-4	-3	-2

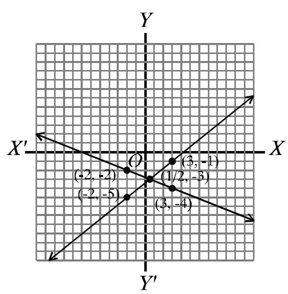
$\therefore$ three points on the graph of the equation are :
$(3,-4),\left(\frac{1}{2},-3\right),(-2,-2)$.

Again, from equation (2), $5y=4x-17$, or $y=\frac{4x-17}{5}$

Taking some convenient values of x in the equation (2), we find the corresponding values of y and make the adjoining table :

x	3	$\frac{1}{2}$	-2
y	-1	-3	-5

$\therefore$ three points on the graph of the equation are : $(3,1),\left(\frac{1}{2},-3\right),(-2,-5)$.



Let XOX' and YOY' be x axis and y axis respectively and O, the origin.
We take each two sides of the smallest squares as unit along with both axes.

Now, we plot the points $(3,-4),\left(\frac{1}{2},-3\right)$ and $(-2,-2)$ obtained from equation (1) in the graph paper and join them each other. The graph is a straight line. Similarly, we plot the points $(3,-1),\left(\frac{1}{2},-3\right),(-2,-5)$ obtained from equation (2) and join them each other. The graph is a straight line.

Let the straight lines intersect at P. It is seen from the graph, coordinates of P are $\left(\frac{1}{2},-3\right)$.

$\therefore$ solution is $(x, y)=\left(\frac{1}{2},-3\right)$

Example 10. Solve with the help of graphs : $3-\frac{3}{2}x=8-4x$

Solution : Given equation is $3-\frac{3}{2}x=8-4x$

Let, $y=3-\frac{3}{2}x=8-4x$

$\therefore \quad y=3-\frac{3}{2}x$.........(1)

And, $y=8-4x$..........(2)

Now, Taking some values of x in equation (1), we find the corresponding values of y and make the adjoining table :

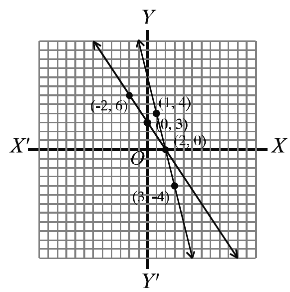
x	-2	0	2
y	6	3	0

$\therefore$ three points on the graph of the equation (1) are : $(-2,6),(0,3),(2,0)$

Again, taking some values of x in equation (2), we find the corresponding values of y and make the adjoining table :

x	1	2	3
y	4	0	-4

$\therefore$ three points on the graph of the equation (2) are : $(1,4),(2,0),(3,-4)$



Let XOX' and YOY' be x axis, y axis respectively and O, the origin. We take each side of the smallest squares along with both axes as unit. Now, we plot the points $(-2,6),(0,3),(2,0)$, obtained from equation (1) on the graph paper and join them each other. The graph is a straight line. In the same way, we plot the points $(1,4),(2,0),(3,-4)$ obtained from equation (2) and join them each other. This graph is also a straight line. Let the two straight lines intersect at P. It is seen from the picture that the coordinates of the point of intersection are $(2,0)$.

$\therefore$ solution is $x=2$, or solution is 2

> **Activity :** Find four points on the graph of the equation $2x-y-3=0$ in terms of a table. Then, taking unit of a fixed length on the graph paper, plot the points and join them each other. Is the graph a straight line ?

Exercise 12·3

Solve by graphs :

1. $3x+4y=14$, $4x-3y=2$
2. $2x-y=1$, $5x+y=13$
3. $2x+5y=1$, $x+3y=2$
4. $3x-2y=2$, $5x-3y=5$
5. $\frac{x}{2}+\frac{y}{3}=2$, $2x+3y=13$
6. $3x+y=6$, $5x+3y=12$

7. $3x + 2y = 4$
$3x - 4y = 1$

8. $\frac{x}{2} + \frac{y}{3} = 3$
$x + \frac{y}{6} = 3$

9. $3x + 2 = x - 2$

10. $3x - 7 = 3 - 2x$

12·5 Formation of simultaneous equations from real life problems and solution.

In everyday life, there occur some such mathematical problems which are easier to solve by forming equations. For this, from the condition or conditions of the problem, two mathematical symbols, mostly the variables x, y are assumed for two unknown expressions. Two equations are to be formed for determining the values of those unknown expressions. If the two equations thus formed are solved, values of he unknown quantities will be found.

Example 11. If 5 is added to the sum of the two digits of a number consisting of two digits, the sum will be three times the digits of the tens place. Moreover, if the places of the digits are interchanged, the number thus found will be 9 less than the original number. Find the number.

Solution : Let the digit of the tens place of the required number be x and its digits of the units place is y. Therefore, the number is $10x + y$.

$\therefore$ by the 1st condition, $x + y + 5 = 3x$.........(1)

and by the 2nd condition, $10y + x = (10x + y) - 9$.......(2)

From equation (1), we get, $y = 3x - x - 5$, or $y = 2x - 5$........(3)

Again from equations (2), we get,

$10y - y + x - 10x + 9 = 0$

or $9y - 9x + 9 = 0$

or $y - x + 1 = 0$

or $2x - 5 - x + 1 = 0$ [putting the value of y from (3)]

or $x = 4$

putting the value of x in (3), we get,

$y = 2 \times 4 - 5$

$= 8 - 5$

$= 3$

$\therefore$ the number will be

$10x + y = 10 \times 4 + 3$

$= 40 + 3$

$= 43$

$\therefore$ the number is 43

Example 12. 8 years ago, father's age was eight times the age of his son. After 10 years, father's age will be twice the age of the son. What are their present ages ?

Solution : Let the present age of father be x year and age of son is y year.

$\therefore$ by 1st condition $x - 8 = 8(y - 8)$........(1)

and by 2nd condition, $x+10=2(y+10)$.......(2)

From (1), we get, $x-8=8y-64$

or $x=8y-64+8$

or $x=8y-56$............(3)

From (2), we get, $x+10=2y+20$

or $8y-56+10=2y+20$ [putting the value of x from (3)]

or $8y-2y=20+56-10$

or $6y=66$

or $y=11$

$\therefore$ from (3), we get, $x=8\times11-56=88-56=32$

$\therefore$ at present, Father's age is 32 years and son's age is 11 years.

Example 13. Twice the breadth of a rectangular garden is 10 metres more than its length and perimeter of the garden is 100 metre.

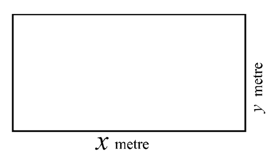
a. Assuming the length of the garden to be x metre and its breadth to be y metre, form system of simultaneous equations.

b. Find the length and breadth of the garden.

c. There is a path of width 2 metres around the outside boundary of the garden. To make the path by bricks, it costs $110\cdot00$ per square metre. What will be the total cost ?

Solution : a. Length of the rectangular garden is x metre and its breadth is y metre.

$\therefore$ by 1st condition, $2y=x+10$........(1)



and by 2nd condition, $2(x+y)=100$.......(2)

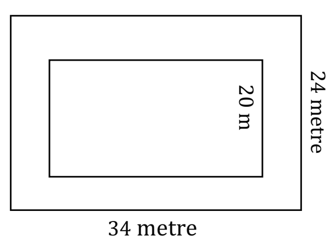
b. From equation (1), we get, $2y=x+10$........(1)

From equation (2), we get, $2x+2y=100$.......(2)

or $2x+x+10=100$ [putting the value of $2y$ from (1)]

or $3x=90$

or $x=30$



$\therefore$ from (1), we get, $2y=30+10$ [putting the value of x]

or, $2y=40$

or, $y=20$

$\therefore$ length of the garden is 30 metres and its breadth is 20 metres.

c. Length of the garden with the path is (30+4) metres.

= 34 metres

and breadth is (20+4) metres = 24 metres.

$\therefore$ Area of the path = Area of the graden with the path − area of the garden

$= (34\times 24 - 30\times 20)$ square metre

$= (816 - 600)$ square metre

$= 216$ square metre

$\therefore$ cost for making the path by bricks

= Tk. 216×110

= Tk. 23760

Activity : If in triangle ABC, $\angle B = 2x$ degree, $\angle C = x$ degree, $\angle A = y$ degree and $\angle A = \angle B + \angle C$, find the value of x and y.	

Exercise 12·4

1. For which of the following conditions are the system of equations $ax+by+c=0$ and $px+qy+r=0$ consistent and mutually independent ?

   a. $\frac{a}{p}\neq\frac{b}{q}$ b. $\frac{a}{p}=\frac{b}{q}=\frac{c}{r}$ c. $\frac{a}{p}=\frac{b}{q}\neq\frac{c}{r}$ d. $\frac{a}{p}=\frac{b}{q}$

2. If $x+y=4, x-y=2$, which one of the following is the value of (x, y) ?

   a. $(2, 4)$ b. $(4, 2)$ c. $(3,1)$ d. $(1, 3)$

3. If $x+y=6$ and $2x=4$, what is the value of y ?

   a. 2 b. 4 c. 6 d. 8

4. For which one of the following equation is the adjoining chart correct?

   a. $y=x-4$ b. $y=8-x$

   c. $y=4-2x$ d. $y=2x-4$

x	0	2	4
7	-4	0	4

5. If $2x-y=8$ and $x-2y=4$, what is $x+y$?

   a. 0 b. 4 c. 8 d. 12

6. Observe the following information : :

   i. The equations $2x-y=0$ and $x-2y=0$ are mutually dependent.

   ii. Graph of the equation $x-2y+3=0$ passes through the point $(-3, 0)$.

   iii. Graph of the equation $3x+4y=1$ is a straight line.

On the basis of information above, which one of the following is correct ?

   a. i and ii b. ii and iii c. i and iii d. i, ii and iii

7. Length of the floor of a rectangular r oom is 2 metres more than its breadth and perimeter of the floor is 20 metres.

Answer the following questions :

(1) What is the length of the floor of the room in metre ?

a. 10 b. 8 c. 6 d. 4

(2) What is the area of the floor of the room in square metre ?

a. 24 b. 32 c. 48 d. 80

(3) How much taka will be the total cost for decorating the floor with mosaic at Tk. 900 per square metre ?

a. 72000 b. 43200 c. 28800 d. 21600

Solve by forming simultaneous equations (8 - 15) :

8. If 1 is added to each of numerator and denominator of a fraction, the fraction will be $\frac{4}{5}$. Again, if 5 is subtracted from each of numerator and denominator, the fraction will be $\frac{1}{2}$. Find the fraction.
9. If 1 is subtracted from numerator and 2 is added to denaminator of a fraction, the fraction will be $\frac{1}{2}$. Again, if 7 is subtracted from numerator and 2 is subtracted from denominator, the fraction will be $\frac{1}{3}$. Find the fraction.
10. The digit of the units place of a number consisting of two digits is 1 more than three times the digit of tens place. But if the places of the digits are interchanged, the number thus found will be equal to eight times the sum of the digits. What is the number ?
11. Difference of the digits of a number consisting of two digits is 4. If the places of the digits are interchanged, sum of the numbers so found and the original number will be 110 ; find the number.
12. Present age of mother is four times the sum of the ages of her two daughters. After 5 years, mother's age will be twice the sum of the ages of the two daughters. What is the present age of the mother ?
13. If the length of a rectangular region is decreased by 5 metres and breadth is increased by 3 metres, the area will be less by 9 square metres. Again, if the length is increased by 3 metres and breadth is increased by 2 metres, the area will be increasd by 67 square metres. Find the length and breadth of the rectangle.

14. A boat, rowing in favour of current, goes 15 km per hour and rowing against the current goes 5 km per hour. Find the speed of the boat and current.

15. A labourer of a garments serves on the basis of monthly salary. At the end of every year she gets a fixed increment. Her monthly salary beomes Tk. 460 after 4 years and Tk. 6 after 8 years. Find the salary at the beginning of her service and amount of annual increment of salary.

16. A system of simple equations are

$$x + y = 10$$
$$3x - 2y = 0$$

a. Show that the equations are consistent. How many solutions do they have ?

b. Solving the system of equations, find (x, y).

c. Find the area of the triangle formed by the straight lines indicated by the equations with the x axis.

17. If 7 is added to the num. of a fraction, the value of the fraction is the integer 2. Again, if 2 is subtracted from the denominator, value of the fraction is the integer 1.

a. Form a system of equations by the fraction $\frac{x}{y}$.

b. Find (x, y) by solving the system of equations by the method of cross-multiplication. What is the fraction ?

c. Draw the graphs of the system of equations and verify the correctness of the obtained values of (x, y).

Chapter Thirteen

Finite Series

The term 'order' is widely used in our day to day life. Such as, the concept of order is used to arrange the commodities in the shops, to arrange the incidents of drama and ceremony, to keep the commodities in attractive way in the godown. Again, to make many works easier and attractive, we use large to small, child to old, light to originated heavy etc. types of order. Mathematical series have been of all these concepts of order. In this chapter, the relation between sequence and series and contents related to them have been presented.

At the end of this chapter, the students will be able to –

- Describe the sequence and series and determine the difference between them
- Explain finite series
- Form formulae for determining the fixed term of the series and the sum of fixed numbers of terms and solve mathematical problems by applying the formulae
- Determine the sum of squares and cubes of natural numbers
- Solve mathematical problems by applying different formulae of series
- Construct formulae to find the fixed term of a geometrical progression and sum of fixed numbers of terms and solve mathematical problems by applying the formulae.

Sequence

Let us note the following relation :

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots\cdots\cdots & n & \cdots\cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 2 & 4 & 6 & 8 & 10 & \cdots\cdots\cdots & 2n & \cdots\cdots\cdots \end{array}$$

Here, every natural number n is related to twice the number $2n$. That is, the set of positive even numbers $\{2, 4, 6, 8, ...\}$ is obtained by a method from the set of natural numbers $N = \{1, 2, 3, ...\}$. This arranged set of even number is a sequence. Hence, some quantities are arranged in a particular way such that the antecedent and subsequent terms becomes related. The set of arranged quantities is called a sequence.

The aforesaid relation is called a function and defined as $f(n) = 2n$. The general term of this sequence is $2n$. The terms of any sequence are infinite. The way of writing the sequence with the help of general term is $< 2n >, n = 1, 2, 3, \ldots \ldots$ or, $\{2n\}_{n=1}^{+\infty}$ or, $<2n>$

The first quantity of the sequence is called the first term, the second quantity is called second term, the third quantity is called the third term etc. The first term of the sequence 1, 3, 5, 7, ... is 1, the second term is 2 etc.

Followings are the four examples of sequence :

$1,\quad 2,\quad 3, \ldots\ldots\ldots\ldots\ldots\ldots, n, \ldots\ldots\ldots\ldots\ldots\ldots$

$1,\quad 3,\quad 5, \ldots\ldots\ldots\ldots\ldots\ldots, (2n-1), \ldots\ldots\ldots\ldots$

$1,\quad 4,\quad 9, \ldots\ldots\ldots\ldots\ldots\ldots, n^2, \ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{1}{2},\quad \frac{2}{3},\quad \frac{3}{4}, \ldots\ldots\ldots\ldots\ldots\ldots, \frac{n}{n+1}, \ldots\ldots\ldots\ldots\ldots\ldots$

> **Activity :** 1. General terms of the six sequences are given below. Write down the sequences :
>
> $(i)\ \frac{1}{n}\quad (ii)\ \frac{n-1}{n+1}\quad (iii)\ \frac{1}{2^n}\quad (iv)\ \frac{1}{2^{n-1}}\quad (v)\ (-1)^{n+1}\frac{n}{n+1}\quad (vi)\ (-1)^{n-1}\frac{n}{2n+1}$.
>
> 2. Each of you write a general term and then write the sequence.

Series

If the terms of a sequence are connected successively by + sign, a series is obtained. Such as, $1+3+5+7+.......$ is a series. The difference between two successive terms of the series is equal. Again, $2+4+8+16+........$ is a series. The ratio of two successive terms is equal. Hence, the characterstic of any series depends upon the relation between its two successive terms. Among the series, two important series are arithmetic series and geometric series.

Arithmetic series

If the difference between any term and its antecedent term is always equal, the series is called arithmetic series.

Example : $1+3+5+7+9+11$ is a series. The first term of the series is 1, the second term is 3, the third term is 5, etc.

Here, second term – first term = $3-1=2$, third term – second term = $5-3=2$, fourth term – third term = $7-5=2$, fifth term – fourth term = $9-7=2$, sixth term – fifth term = $11-9=2$.

Hence the series is an arithmetic series. In this series, the difference between two terms is called common difference. The common difference of the mentioned series is 2. The numbers of terms of the series are fixed. That is why the series is finite series. It is to be noted that if the terms of the series are not fixed, the series is called infinite series, such as, $1+4+7+10+\ldots\ldots$ is an infinite series. In an arithmetic series, the first term and the common difference are generally denoted by a and d respectively. Then by definition, if the first term is a, the second term is $a+d$, the third term is $a+2d$, etc. Hence, the series will be $a+(a+d)+(a+2d)+\ldots$

Determination of common term of an arithmetic series

Let the first term of arithmetic series be a and the common difference be d, terms of the series are :

first term $= a \quad = a + (1-1)\,d$

second term $= a + d \quad = a + (2-1)\,d$

third term $= a + 2d \quad = a + (3-1)\,d$

forth term $= a + 3d \quad = a + (4-1)\,d$

....

....

$\therefore$ nth term $= a + (n-1)d$

This nth term is called common term of arithmetic series. If the first term of an arithmetic series in a and common difference is d, all the terms of the series are determined successively by putting $n = 1, 2, 3, 4, \ldots\ldots$ in the nth term.

Let the first term of an arithmetic series be 3 and the common difference be 2. Then second term of the series $= 3 + 2 = 5$, third term $= 3 + 2 \times 2 = 7$, forth term $= 3 + 3 \times 2 = 9$ etc.

Therefore, nth term of the series $= 3 + (n-1) \times 2 = 2n + 1$.

Activity : If the first term of an arithmetic series is 5 and common difference is 7, find the first six terms, 22nd term, r th term and $(2p+1)$th term.

Example 1. Of the series, $5 + 8 + 11 + 14 + \cdots\cdots$ which term is 383 ?

Solution : The first term of the series $a = 5$, common difference $d = 8 - 5 = 11 - 8 = 3$

$\therefore$ It is an arithmetic series.

Let, nth term of the series $= 383$

We know that, nth term $= a + (n-1)d$.

$\therefore \quad a + (n-1)d = 383$

or, $5 + (n-1)3 = 383$

or, $5 + 3n - 3 = 383$

or, $3n = 383 - 5 + 3$

or, $3n = 381$

or, $n = \dfrac{381}{3}$

$\therefore \quad n = 127$

$\therefore$ 127^{th} term of the given series $= 383$.

Sum of *n* terms of an Arithmetic series

Let the first term of any arithmetic series be a, last term be p, common difference be d, number of terms be n and sum of n numbers of terms be S_n.

Writing from the first term and conversely from the last term of the series we get,

$$S_n = a + (a+d) + (a+2d) + \cdots \cdots + (p-2d) + (p-d) + p \qquad (i)$$

$$\text{and} \quad S_n = p + (p-d) + (p-2d) + \cdots \cdots + (a+2d) + (a+d) + a \qquad (ii)$$

Adding, $2S_n = (a+p) + (a+p) + (a+p) + \ldots\ldots + (a+p) + (a+p) + (a+p)$

or, $2S_n = n(a+p)$ [$\because$ number of terms of the series is n]

$\therefore \quad S_n = \frac{n}{2}(a+p)$ $\qquad (iii)$

Again, nth term $= p = a + (n-1)d$. Putting this value of p in (iii) we get,

$$S_n = \frac{n}{2}[a + \{a + (n-1)d\}]$$

i.e., $S_n = \frac{n}{2}\{2a + (n-1)d\}$ $\qquad (iv)$

If the first term of arithmetic series a, last term p and number of terms n are known, the sum of the series can be determined by the formula (iii). But if first term the a, common difference d, number of terms n are known, the sum of the series are determined by the formula (iv).

Determination of the sum of first *n* terms of natural numbers

Let S_n be the sum of n numbers of natural numbers i.e.

$S_n = 1 + 2 + 3 + \cdots \cdots \cdots + (n-1) + n$ $\qquad (i)$

Writing from the first term and conversely from the last term of the series we get,

$$S_n = 1 + 2 + 3 + \cdots \cdots \cdots + (n-2) + (n-1) + n \qquad (i)$$

and $$S_n = n + (n-1) + (n-2) + \cdots \cdots + 3 + 2 + 1 \qquad (ii)$$

Adding, $2S_n = (n+1) + (n+1) + (n+1) + \cdots \cdots \cdots + (n+1)$ [n number of terms]

or, $2S_n = n(n+1)$

$$\therefore \ S_n = \frac{n(n+1)}{2} \qquad (iii)$$

Example 2. Find the sum total of first 50 terms of natural numbers.

Solution : Using formula (iii) we get,

$$S_{50} = \frac{50\ (50+1)}{2} = 25 \times 51 = 1275$$

$\therefore$ The sum total of first 50 natural numbers is 1275

Example 3. $1 + 2 + 3 + 4 + \cdots \cdots \cdots + 9$ = what ?

Solution : The first term of the series $a = 1$, common difference $d = 2 - 1 = 1$ and the last term $p = 9$

$\therefore$ It is an arithmetic series.

Let the nth term of the series $= 9$

We know, nth term of an arithmetic progression $= a + (n-1)d$

$\therefore \quad a + (n-1)d = 9$

or, $\quad 1 + (n-1)1 = 9$

or, $\quad 1 + n - 1 = 9$

Alternative method :

Since

$S_n = \frac{n}{2}(a + p,)$

$\therefore \ S_9 = \frac{9}{2}(1 + 9\ )$

$\therefore \quad n = 9$

$= \dfrac{9 \times 10}{2} = 45$

From (*iv*) formula, the sum of first n-terms of an arithmetic series

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Hence, the sum 9 terms of the series $S_9 = \dfrac{9}{2}\{2 \times 1 + (9-1) \times 1\} = \dfrac{9}{2}(2+8)$

$$= \frac{9 \times 10}{2} = 9 \times 5 = 45$$

Example 4. What is the sum of 30 terms of the series $7 + 12 + 17 + \cdots\cdots\cdots$

Solution : First term of the series $a = 7$, common difference $d = 12 - 7 = 5$

$\therefore$ It is an arithmetic series. Here, number of terms $n = 30$.

We know that the sum of n-terms of an arithmetic series

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

So, the sum of 30 terms $S_{30} = \dfrac{30}{2}\{2.7 + (30-1)5\} = 15(14 + 29 \times 5)$

$$= 15(14 + 145) = 15 \times 159$$

$$= 2385$$

Example 5. A deposits Tk. 1200 from his salary in the first month and in every month of subsequent months, he deposits Tk. 100 more than the previous months.

(i) How much does he deposit in nth month ?

(ii) Express the aforesaid problem in series upto n terms.

(iii) How much does he deposit in first n-months ?

(iv) How much does he deposit in a year ?

Solution : (i) In the first month, he deposits Tk. 1200

In the second month, he deposits Tk. $(1200 + 100) =$ Tk. 1300

In third month, he deposits Tk. $(1300 + 100) =$ Tk. 1400

In forth month, he deposits Tk. $(1400 + 100) =$ Tk. 1500

Hence, it is an arithmetic series whose first term is $a = 1200$, common difference $d = 1300 - 1200 = 100$.

nth term of the series $= a + (n-1)d$

$$= 1200 + (n-1)100 = 1200 + 100n - 100$$

$$= 100n + 1100$$

Therefore, he deposits Tk. $(100n + 1100)$ in nth month.

(ii) The series, in this case upto n-numbers of terms will be

$1200 + 1300 + 1400 + \cdots\cdots\cdots + (100n + 1100)$

(iii) In n numbers of months, he deposits Tk. $\frac{n}{2}\{2a + (n-1)d\}$

$= \text{Tk. } \frac{n}{2}\{2 \times 1200 + (n-1)100\}$

$= \text{Tk. } \frac{n}{2}(2400 + 100n - 100) = \text{Tk. } \frac{n}{2} \times 2(1150 + 50n)$

$= \text{Tk. } n(50n + 1150)$

(iv) We know that 1 year = 12 months. Here $n = 12$.

Therefore, A deposits in 1 year $= \text{Tk. } 12(50 \times 12 + 1150)$

$= \text{Tk. } 12(600 + 1150)$

$= \text{Tk. } 12 \times 1750$

$= \text{Tk. } 21000$.

Exercise 13.1

1. Find the common difference and the 12th terms of the series $2 - 5 - 12 - 19 - \cdots$
2. Which term of the series $8 + 11 + 14 + 17 + \cdots\cdots\cdots$ is 392 ?
3. Which term of the series $4 + 7 + 10 + 13 + \cdots\cdots\cdots$ is 301 ?
4. If the pth term of an arithmetic series is p^2 and qth term is q^2, what is $(p+q)$th term of the series ?
5. If the mth tem of an arithmetic series is n and nth term is m, what is $(m+n)$th term of the series ?
6. What is the number of n terms of the series $1 + 3 + 5 + 7 + \cdots\cdots\cdots$?
7. What is the sum of first 9 terms of the series $8 + 16 + 24 + \cdots\cdots\cdots n$?
8. $5 + 11 + 17 + 23 + \cdots\cdots\cdots + 59 =$ What ?
9. $29 + 25 + 21 + \cdots\cdots\cdots - 23 =$ What ?
10. The 12th term of an arithmetic series is 77. What is the sum of the first 23 terms?
11. If the 16th term of an arithmetic series is -20, what will be the sum of first 31 terms ?
12. The total sum of first n terms of the series $9 + 7 + 5 + \cdots\cdots\cdots$ is -144. Find the value of n.
13. If the sum of first n terms of the series $2 + 4 + 6 + 8 + \cdots\cdots\cdots$ is 2550, find the value of n.
14. If the sum of first n terms of the series is $n(n+1)$, find the series.
15. If the sum of first n terms of the series is $n(n+1)$, what is the sum of first 10 terms ?
16. If the sum of 12 terms of an arithmetic series is 144 and first 20 terms is 560, find the sum of first 6 terms.

17. The sum of the first m terms of an arithmetic series is n and the first n terms is m. Find the sum of first $(m+n)$ terms.

18. If the pth, qth and rth terms of an arithmetic series are a, b, c, respectively, show that $a(q-r)+b(r-p)+c(p-q)=0$.

19. Show that, $1+3+5+7+\cdots\cdots\cdots+125 = 169+171+173+\cdots\cdots\cdots+209$.

20. A man agrees to refund the loan of Tk. 2500 in some parts. Each part is Tk. 2 more than the previous part. If the first part is Tk. 1, in how many parts will the man be able to refund that amount ?

Determinaton of the sum of Squares of the first *n* numbers of Natural Numbers

Let S_n be the number of squares of the first n numbers of natural numbers

i.e., $S_n = 1^2+2^2+3^2+\cdots\cdots\cdots+n^2$

We know,

$$r^3-3r^2+3r-1=(r-1)^3$$

$$\text{or, } r^3-(r-1)^3 = 3r^2-3r+1$$

In the above identity, putting, $r = 1, 2, 3, \ldots\ldots, n$ we get,

$$1^3-0^3 = 3.1^2-3.1+1$$
$$2^3-1^3 = 3.2^2-3.2+1$$
$$3^3-2^3 = 3.3^2-3.3+1$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$n^3-(n-1)^3 = 3.n^2-3.n+1$$

Adding, we get,

$$n^3-0^3 = 3(1^2+2^2+3^2+\cdots\cdots\cdots+n^2)-3(1+2+3+\cdots\cdots\cdots+n)+(1+1+1+\cdots\cdots\cdots+1)$$

$$\text{or, } n^3 = 3S_n - 3.\frac{n(n+1)}{2}+n \quad \left[\because 1+2+3+\cdots\cdots\cdots+n = \frac{n(n+1)}{2}\right]$$

$$\text{or, } 3S_n = n^3+\frac{3n(n+1)}{2}-n$$

$$= \frac{2n^3+3n^2+3n-2n}{2} = \frac{2n^3+3n^2+n}{2} = \frac{n(2n^2+3n+1)}{2}$$

$$= \frac{n(2n^2+2n+n+1)}{2} = \frac{n\{2n(n+1)+1(n+1)\}}{2}$$

$$\text{or, } 3S_n = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6}$$

The sum of cubes of the first *n* numbers of Natural Numbers

Let S_n be the sum of cubes of the first n numbers of natural numbers.

That is, $S_n = 1^3 + 2^3 + 3^3 + \cdots\cdots\cdots + n^3$

We know that, $(r+1)^2 - (r-1)^2 = (r^2 + 2r + 1) - (r^2 - 2r + 1) = 4r.$

or, $(r+1)^2 r^2 - r^2 (r-1)^2 = 4r.r^2 = 4r^3$ [Multiplying both the sides by r^2]

In the above identity, putting $r = 1, 2, 3, \ldots\ldots, n$

We get,

$2^2.1^2 - 1^2.0^2 = 4.1^3$

$3^2.2^2 - 2^2.1^2 = 4.2^3$

$4^2.3^2 - 3^2.3^2 = 4.3^3$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$(n+1)^2 n^2 - n^2 (n-1)^2 = 4n^3$

Adding, we get, $(n+1)^2.n^2 - 1^2.0^2 = 4(1^3 + 2^3 + 3^3 + \cdots\cdots\cdots + n^3)$

or, $(n+1)^2.n^2 = 4S_n$

or, $S_n = \dfrac{n^2(n+1)^2}{4}$

$\therefore \quad S_n = \left\{\dfrac{n(n+1)}{2}\right\}^2$

Necessary formulae :

$$1 + 2 + 3 + \cdots\cdots\cdots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots\cdots\cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots\cdots\cdots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

N.B. $1^3 + 2^3 + 3^3 + \cdots\cdots\cdots + n^3 = (1 + 2 + 3 + \cdots\cdots\cdots + n)^2.$

Activity : 1. Find the sum of natural even numbers of the first n numbers.

2. Find the sum of squares of natural odd numbers of the first n numbers.

Geometric series

If the ratio of any term and its antecedent term of any series is always equal i.e., if any term divided by its antecedent term, the quotient is always equal, the series is called a geometric series and the quotient is called common ratio. Such as, of the series $2+4+8+16+32$, the first term is 2, the second term is 4, the third term is 8, the fourth term is 16 and the fifth term is 32. Here, the ratio of the second term to the first term = $\frac{4}{2}=2$, the ratio of the third term to the second term = $\frac{8}{4}=2$, the ratio of the fourth term to the third term = $\frac{16}{8}=2$, the ratio of the fifth term to the fourth term = $\frac{32}{16}=2$.

In this series, the ratio of any term to its antecedent term is always equal. The common ratio of the mentioned series is 2. The numbers of terms of the series are fixed. That is why the series is finite geometric series. The geometric series is widely used in different areas of physical and biological science, in organizations like Bank and Life Insurance etc, and in different branches of technology. If the numbers of terms are not fixed in a geometric series, it is called an infinite geometric series.

The first term of a geometric series is generally expressed by a and common ratios by r. So by definition, if the first term is a, the second term is ar, the third term is ar^2, etc. Hence the series will be $a+ar+ar^2+ar^3+\cdots\cdots\cdots$

Activity : Write down the geometric series in the following cases :

(*i*) The first term 4, common ratio 10 (*ii*) The first term 9, common ratio $\frac{1}{3}$

(*iii*) The first term 7, common ratio $\frac{1}{10}$ (*iv*) The first term 3, common ratio 1

(*v*) The first term 1, common ratio $-\frac{1}{2}$ (*vi*) The first term 3, common ratio -1

General term of a Geometric series

Let the first term of a geometric series be a, and common ratio be r. Then, of the series,

first term $=a=ar^{1-1}$, second term $=ar=ar^{2-1}$

third term $=ar^2=ar^{3-1}$, fourth term $=ar^3=ar^{4-1}$

... ...

... ...

nth term = ar^{n-1}

This nth term is called the general term of the geometric series. If the first term of a geometric series a and the common ratio r are known, any term of the series can be determined by putting $r=1, 2, 3,\ldots\ldots$ etc. successively in the nth term.

Example 6. What is the 10th term of the series $2+4+8+16+\cdots\cdots$?

Solution : The first term of the series $a = 2$, common ratio $r = \frac{4}{2} = 2$.

$\therefore$ The given series is a geometric series.

We know that the nth term of geometric series = ar^{n-1}

$\therefore$ 10th term of the series $= 2 \times 2^{10-1}$

$= 2 \times 2^9 = 1024$

Example 7. What is the general term of the series $128 + 64 + 32 + \cdots\cdots\cdots$?

Solution : The first term of the series $a = 128$, common ratio $r = \frac{64}{128} = \frac{1}{2}$.

$\therefore$ It is a geometric series.

We know that the general term of the series = ar^{n-1}

Hence, the general term of the series = $128 \times \left(\frac{1}{2}\right)^{n-1} = \frac{2^7}{2^{n-1}} = \frac{1}{2^{n-1-7}} = \frac{1}{2^{n-8}}$.

Example 8. The first and the second terms of a geometric series are 27 and 9. Find the 5th and the 10th terms of the series.

Solution : The first term of the given series $a = 27$, the second term is 9.

Then the common ratio $r = \frac{9}{27} = \frac{1}{3}$.

$\therefore$ The 5th term = $ar^{5-1} = 27 \times \left(\frac{1}{3}\right)^4 = \frac{27 \times 1}{27 \times 3} = \frac{1}{3}$

and the 10th term = $ar^{10-1} = 27 \times \left(\frac{1}{3}\right)^9 = \frac{3^3}{3^3 \times 3^6} = \frac{1}{3^6} = \frac{1}{729}$.

Determination of the sum of a Geometric series

Let the first term of the geometric series be a, common ratio r and number of terms n. If S_n is the sum of n terms,

$$S_n = a + ar + ar^2 + \cdots\cdots\cdots + ar^{n-2} + ar^{n-1} \quad (i)$$

and $$r.S_n = ar + ar^2 + ar^3 + \cdots\cdots\cdots + ar^{n-1} + ar^n \quad \text{[multiplying } (i) \text{ by } r] \quad (ii)$$

Subtracting, $S_n - rS_n = a - ar^n$

or, $S_n(1 - r) = a(1 - r^n)$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r}, \text{ when } r < 1$$

Again, subtracting (ii) form (i) we get,

$rS_n - S_n = ar^n - a$ or, $S_n(r - 1) = a(r^n - 1)$

i.e., $$S_n = \frac{a(r^n - 1)}{(r - 1)}, \text{ when } r > 1.$$

Observe : If common ratio is $r = 1$, each term $= a$

Hence, in this case $S_n = a + a + a + \cdots \cdots$ upto n.
$= an.$

Activity : A employed a man from the first April for taking his son to school and taking back home for a month. His wages were fixed to be one paisa in first day, twice of the first day in second day i.e. two paisa, twice of the second day in the third day i.e. four paisa. If the wages were paid in this way, how much would he get after one month including holidays of the week ?

Example 9. What is the sum of the series $12 + 24 + 48 + \cdots \cdots \cdots + 768$?

Solution : The first term of the series is $a = 12$, common ratio $r = \frac{24}{12} = 2 > 1$.

$\therefore$ the series is a geometric series.

Let the nth term of the series $= 768$

We know, nth term $= ar^{n-1}$

$\therefore \quad ar^{n-1} = 768$

or, $\quad 12 \times 2^{n-1} = 768$

or, $\quad 2^{n-1} = \frac{768}{12} = 64$

or, $\quad 2^{n-1} = 2^6$

or, $\quad n - 1 = 6$

$\therefore \quad n = 7.$

Therefore, the sum of the series $= \frac{a(r^n - 1)}{(r-1)}$, when $r > 1$

$$= \frac{12\,(2^7 - 1)}{2-1} = 12 \times (128 - 1) = 12 \times 127 = 1524.$$

Example 10. Find the sum of first eight terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \cdots \cdots$

Solution : The 1st term of the series is $a = 1$, common ratio $r = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$

$\therefore$ It is a geomeric series.

Here the number of terms $n = 8$.

We know, sum of n terms of a geometric series

$$S_n = \frac{a(1 - r^n)}{1 - r}, \text{ when } r < 1.$$

Hence, sum of eight terms of the series is $S_8 = \dfrac{1 \times \left\{1 - \left(\frac{1}{2}\right)^8\right\}}{1 - \frac{1}{2}} = \dfrac{1 - \frac{1}{256}}{\frac{1}{2}}$

$$= 2\left(\frac{256 - 1}{256}\right) = \frac{255}{128} = 1\frac{127}{128}$$

Exercise 13.2

1. If a, b, c, d are cosecutive four terms of a arithmetic series, which one is correct ?

   (a) $b = \frac{c+d}{2}$ (b) $a = \frac{b+c}{2}$ (c) $c = \frac{b+d}{2}$ (d) $d = \frac{a+c}{2}$

2. (i) If $a + (a+d) + a + 2d) + ...$ the sum of first n is terms of the series is $\frac{n}{2}\{2a + (n-1)d\}$

   (ii) $1 + 2 + 3 + ... + n = \frac{n(n+1)(2n+1)}{6}$

   (iii) $1 + 3 + 5 + ... + (2n-1) = n^2$

   Which one of the followings is correct according to the above statements.

   (a) i and ii (b) i and iii (c) ii and iiii (d) i, ii and iii

   Answer the questions 3 and 4 on the basis of following series :

   $\log 2 + \log 4 + \log 8 + ...$

3. Which one is the common difference of the series ?

   (a) 2 (b) 4 (c) log 2 (d) 2 log 2

4. Which one is the 7th term of the series

   (a) log 32 (b) log 64 (c) log 128 (d) log 256

5. Determine the 8th term of the series $64 + 32 + 16 + 8 + ...$

6. Determine the sum of first fourteen terms of the series $3 + 9 + 27 + ...$

7. Which of the term is $\frac{1}{2}$ of the series $128 + 64 + 32 + ...$

8. If the 5th terms of a geometric series $\frac{2\sqrt{3}}{9}$ and the 10th term are $\frac{8\sqrt{2}}{8}$, find the 3rd term of the series.

9. Which of the term is $8\sqrt{2}$ of the sequence $\frac{1}{\sqrt{2}}, -1, \sqrt{2}, \cdots \cdots \cdots$?

10. If $5 + x + y + 135$ is geometric series, find the value of x and y.

11. If $3 + x + y + z + 243$ is geometric series, find the value of x, y and z.

12. What is the sum of first seven terms of the series $2 - 4 + 8 - 16 + \cdots \cdots \cdots$?

13. Find the sum of $(2n+1)$ terms of the series $1 - 1 + 1 - 1 + \cdots \cdots \cdots$

14. What is the sum of first ten terms of the series $\log 2 + \log 4 + \log 8 + \cdots \cdots \cdots$?

15. Find the sum of first twelve terms of the series

    $\log 2 + \log 16 + \log 128 + \cdots \cdots$?

16. If the sum of n terms of the series $2 + 4 + 8 + 16 + \cdots \cdots \cdots$ is 254, what is the value of n ?

17. What is sum of $(2n+2)$ terms of the series $2-2+2-2+\cdots\cdots\cdots$?
18. If the sum of cubes of n natural numbers is 441, find the value of n and find the sum of those terms.
19. If the sum of cubes of the first n natural numbers is 225, find the value of n and find the sum of square of those terms ?
20. Show that $1^3+2^3+3^3+.............10^3=(1+2+3+.....+10)^2$.
21. If $\dfrac{1^3+2^3+3^3+\cdots\cdots\cdots+n^3}{1+2+3+\cdots\cdots\cdots+n}=210$, what is the value of n ?
22. A iron-bar with length one metre is divided into ten-pieces such that the lengths of the pieces form a geometric progression. If the largest pieces is ten times of the smallest one, find the length in approximate millimetre of the smallest piece.
23. The first term of a geometric series is a, common ratio is r, the fourth term of the series is 2 and the 9th term is $8\sqrt{2}$.

    (a) Express the above information by two equations.

    (b) Find the 12th term of the series.

    (c) Find the series and then determine the sum of the first seven terms of the series.
24. The nth term of The a series is $2n-4$.

    (a) Find the series.

    (b) Find the 10th term of the series and determine the sum of first 20 terms.

    (c) Considering the first term of the obtained series as 1st term and the common difference as common ratio, construct a new series and find the sum of first 8 terms of the series by applying the formula.

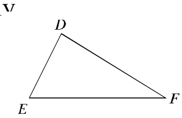
Chapter Fourteen

Ratio, Similarity and Symmetry

For comparing two quantities, their ratios are to be considered. Again, for determining ratios, the two quantities are to be measured in the same units. In algebra we have discussed this in detail.

At the end of this chapter, the students will be able to

- Explain geometric ratios
- Explain the internal division of a line segment
- Verify and prove theorems related to ratios
- Verify and prove theorems related to similarity
- Explain the concepts of symmetry
- Verify line and rotational symmetry of real objects practically



14·1 Properties of ratio and proportion

(i) If $a : b = x : y$ and $c : d = x : y$, it follows that $a : b = c : d$.

(ii) If $a : b = b : a$, it follows that $a = b$

(iii) If $a : b = x : y$, it follows that $b : a = y : x$ (inversendo)

(iv) If $a : b = x : y$, it follows that $a : x = b : y$ (alternendo)

(v) If $a : b = c : d$, it follows that $ad = bc$ (cross multiplication)

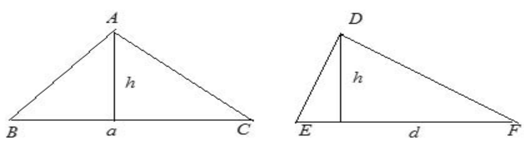
(vi) If $a : b = x : y$, it follows that $a + b : b = x + y : y$ (componendo) and $a - b : b = x - y : y$ (dividendo)

(vii) If $\frac{a}{b} = \frac{c}{d}$, it follows that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo-dividendo)

Geometrical Proportion

Earlier we have learnt to find the area of a triangular region. Two necessary concept of ratio are to be formed from this.

(1) **If the heights of two triangles are equal, their bases and areas are proportional.**



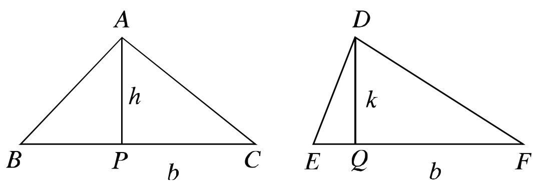
Let the bases of the triangles ABC and DEF be $BC = a$, $EF = d$ respectively and the height in both cases be h.

Hence, the area of the triangle ABC = $\frac{1}{2}a \times h$ and the area of the triangle $DEF = \frac{1}{2}d \times h$.

Therefore, area of the triangle ABC :area of the triangle DEF $= \frac{1}{2}a \times h : \frac{1}{2}d \times h$

$= a : d = BC : EF$ that is, the areas and bases ate proportional.

(2) **If the bases of two triangles are equal, their heights and areas are proportional.**



Let the heights of the triangles ABC and DEF be AP = h, $DQ = k$ respectively and the base in both cases be b. Hence, the area of the triangle $ABC = \frac{1}{2}b \times h$ and the area of the triangle $DEF = \frac{1}{2}b \times k$

Therefore, area of the triangle ABC :area of the triangle DEF $= \frac{1}{2}b \times h : \frac{1}{2}b \times k$

$= h : k = AP : DQ$

Theorem 1

A straight line drawn parallel to one side of a triangle intersects the other two sides or those sides produced proportionally.

Proposition : In the figure, the straight line DE is parallel to the side BC of the triangle ABC. DE intersects AB and AC or their produced sections at D and E respectively. It is required to prove that, $AD : DB = AE : EC$.

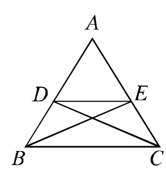


Fig-1

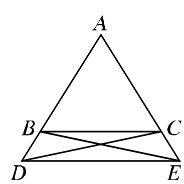


Fig-2

Construction: Join B, E and C, D.

Proof:

Steps	Justification
(1) The heights of ΔADE and ΔBDE are equal.	[The bases of the triangles of equal height are proportional]

$\therefore \dfrac{\Delta ADE}{\Delta BDE} = \dfrac{AD}{DB}$	[The bases of the triangles of equal height are proportional]
(2) Again, The heights of ΔADE and ΔDEC are equal.	[On the same base and between same pair of lines]
$\therefore \dfrac{\Delta ADE}{\Delta DEC} = \dfrac{AE}{EC}$	
(3) But $\Delta BDE = \Delta DEC$	
$\therefore \dfrac{\Delta ADE}{\Delta BDE} = \dfrac{\Delta ADE}{\Delta DEC}$	
(4) Therefore, $\dfrac{AD}{DB} = \dfrac{AE}{EC}$	
i.e., $AD : DB = AE : EC$.	

Corollary 1. If the line parallel to BC of the triangle ABC intersects the sides AB and AC at D and E respectively, $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ and $\dfrac{AB}{BD} = \dfrac{AC}{CE}$.

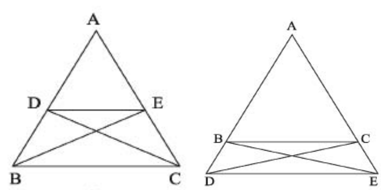
Corollary 2. The line through the mid point of a side of a triangle parallel to another side bisects the third line.

The proposition opposite of theorem 1 is also true. That is, if a line segment divides the two sides of a triangle or the line produced proportionally it is parallel to the third side. Here follows the proof of the theorem.

Theorem 2

If a line segment divides the two sides or their produced sections of a triangle proportionally, it is parallel to the third side.

Proposition : In the triangle ABC the line segment DE divides the two sides AB and AC or their produced sections proportionally. That is, $AD : DB = AE : EC$. It is required to prove that DE and BC are proportional.



Construction: Join B, E and C, D.

Proof:

Steps	Justification
(1) $\dfrac{\Delta ADE}{\Delta BDE} = \dfrac{AD}{DB}$	[Triangles with equal height]
and $\dfrac{\Delta ADE}{\Delta DEC} = \dfrac{AE}{EC}$	[Triangles with equal height]
	[given]
	[From (i) and (ii)]

(2) But $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

(3) Therefore, $\dfrac{\Delta ADE}{\Delta BDE} = \dfrac{\Delta ADE}{\Delta DEC}$

$\therefore \quad \Delta BDE = \Delta DEC$

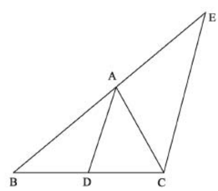
(4) But ΔBDE and ΔDEC are on the same side of the common base *DE*. So they lie between a pair of parallel lines. Hence *BC* and *DE* are parallel.

Theorem 3

The internal bisector of an angle of a triangle divides its opposite side in the ratio of the sides constituting to the angle.

Proposition : In ΔABC the line segment *AD* bisects the internal angle $\angle A$ and intersects the side *BC* at *D*. It is required to prove that $BD : DC = BA : AC$.

Construction: Draw the line segment *CE* parallel to *DA*, so that it intersects the side *BA* produced at *E*.

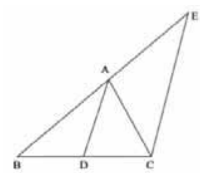


Proof:

Steps	Justification
(1) Since $DA \parallel CE$ and both BC and AC are their transversal	[by construction]
and $\angle AEC = \angle BAD$	[corresponding angles]
and $\angle ACE = \angle CAD$	[alternate angles]
(2) But $\angle BAD = \angle CAD$	[supposition]
$\therefore \angle AEC = \angle ACE$; $\therefore AC = AE$	[Theorem 1]
(3) Again, since $DA \parallel CE$, $\therefore \dfrac{BD}{DC} = \dfrac{BA}{AE}$	
(4) But $AE = AC$ $\therefore \dfrac{BD}{DC} = \dfrac{BA}{AC}$	[step (2)]

Theorem 4

If any side of a triangle is divided internally, the line segment from the point of division to the opposite vertex bisects the angle at the vertex.



Proposition : Let ABC be a triangle and the line segment AD from vertex A divides the side BC at D such that $BD : DC = BA : AC$. It is required to prove that AD bisects $\angle BAC$, i.e. $\angle BAD = \angle CAD$.

Construction : Draw at C the line segment CE parallel to DA, so that it intersects the side BA produced at E.

Proof:

Steps	Justification
(1) In ΔBCE, $DA \parallel CE$	[by construction]
$\therefore BA : AE = BD : DC$	[Theorem 1]
(2) But $BD : DC = BA : AC$	[supposition]
$\therefore BA : AE = BA : AC$	[from steps (1) and (2)]
$\therefore AE = AC$	[Base angles of isosceles are equal]
Therefore, $\angle ACE = \angle AEC$	[corresponding angles]
(4) But $\angle AEC = \angle BAD$	[alternate angles]
and $\angle ACE = \angle CAD$	[from step (2)]
Therefore, $\angle BAD = \angle CAD$	
i.e., the line segment AD bisects $\angle BAC$.	

Exercise 14·1

1. The bisectors of two base angles of a triangle intersect the opposite sides at X and Y respectively. If XY is parallel to the base, prove that the triangle is an isosceles triangle.
2. Prove that if two lines intersect a few parallel lines, the matching sides are proportional.
3. Prove that the diagonals of a trapezium are divided in the same ratio at their point of intersection.
4. Prove that the line segment joining the mid points of oblique sides of a trapezium and two parallel sides are parallel.
5. The medians AD and BE of the triangle ABC intersects each other at G. A line segment is drawn through G parallel to DE which intersects AC at F. Prove that $AC = 6EF$.
6. In the triangle ABC, X is any point on BC and O is a point on AX. Prove that $\Delta AOB : \Delta AOC = BX : XC$
7. In the triangle ABC, the bisector of $\angle A$ intersects BC at D. A line segment drawn parallel to BC intersects AB and AC at E and F respectively. Prove that $BD : DC = BE : CF$.

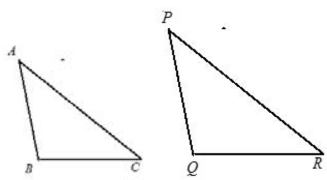
8. If the heights of the equiangular triangles *ABC* and *DEF* are *AM* and *DN* respectively, prove that $AM : DN = AB : DE$.

14·2 Similarity

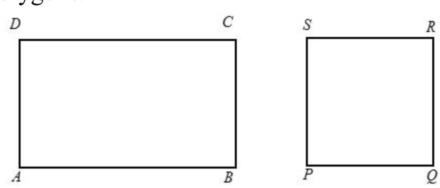
The congruence and similarity of triangles have been discussed earlier in class VII. In general, congruence is a special case of similarity. If two figures are congruent, they are similar; but two similar triangles are not always congruent.

Equiangular Polygons:

If the angles of two polygons with equal number of sides are sequentially equal, the polygons are known as equiangular polygons.



Equiangular triangle



Equiangular quadrilateral

Similar Polygons:

If the vertices of two polygons with equal number of sides can be matched in such a sequential way that

(i) The matching angles are equal

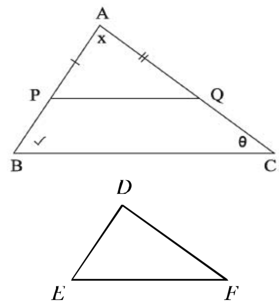
(ii) The ratios of matching sides are equal, the two polygons are called similar polygons.

In the above figures, the rectangle *ABCD* and the square *PQRS* are equiangular since the number of sides in both the figures is 4 and the angles of the rectangle are sequentially equal to the angles of the square (all right angles). Though the similar angles of the figure are equal, the ratios of the matching sides are not the same. Hence the figures are not similar. In case of triangles, situation like this does not arise. As a result of matching the vertices of triangles, if one of the conditions of similarity is true, the other condition automatically becomes true and the triangles are similar. That is, similar triangles are always equiangular and equiangular triangles are always similar.

If two triangles are equiangular and one of their matching pairs is equal, the triangles are congruent. The ratio of the matching sides of two equiangular triangles is a constant. Proofs of the related theorems are given below.

Theorem 5

If two triangles are equiangular, their matching sides are proportional.



Proposition : Let ABC and DEF be triangles with $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. We need to prove that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Construction: Consider the matching sides of the triangles ABC and DEF unequal. Take two points P and Q on AB and AC respectively so that $AP = DE$ and $AQ = DF$. Join P and Q and complete the construction.

Proof:

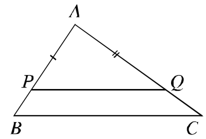
Steps	Justification
(1) In the triangles APQ and DEF $AP = DE$, $AQ = DF$, $\angle A = \angle D$ Therefore, $\Delta APQ \cong \Delta DEF$	[SAS theorem]
Hence, $\angle APQ = \angle DEF = \angle ABC$ and $\angle AQP = \angle DFE = \angle ACB$. That is, the corresponding angles produced as a result of intersections of AB and AC by the line segment PQ are equal. Therefore, $PQ \parallel BC$; $\therefore \frac{AB}{AP} = \frac{AC}{AQ}$ or, $\frac{AB}{DE} = \frac{AC}{DF}$.	[theorem 1]
(2) Similarly, cutting line segments ED and EF from BA and BC respectively, it can be shown that i.e., $\frac{BA}{ED} = \frac{BCEF}{}$ i.e., $\frac{AB}{DE} = \frac{BC}{EF}$; $\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.	[theorem 1]

The proposition opposite of theorem 5 is also true.

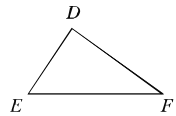
Theorem 6

If the sides of two triangles are proportional, the opposite angles of their matching sides are equal.

Proposition : Let in ΔABC and ΔDEF,
$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.
It is to prove that,
$\angle A = \angle D, \ \angle B = \angle E, \ \angle C = \angle F$.



Construction: Consider the matching sides of the triangles *ABC* and *DEF* unequal. Take two points P and *Q* on *AB* and *AC* respectively so that *AP* = *DE* and *AQ* = *DF*. Join *P* and *Q*.



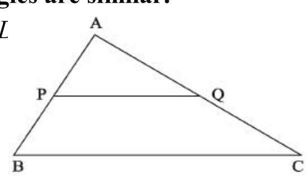
Proof:

Steps	Justification
(1) Since $\frac{AB}{DE} = \frac{AC}{DF}$, so, $\frac{AB}{AP} = \frac{AC}{AQ}$.	
Therefore, $PQ \parallel BC$	[Theorem 2]
$\therefore \ \angle ABC = \angle APQ$ and $\angle ACB = \angle AQP$	[Corresponding angles made by the transversal AB]
$\therefore$ Triangles *ABC* and *APQ* are equiangular.	[Corresponding angles made by the transversal AC]
Therefore, $\frac{AB}{AP} = \frac{BC}{PQ}$, so, $\frac{AB}{DE} = \frac{BC}{AQ}$.	[Theorem 5]
$\therefore \ \frac{BC}{EF} = \frac{BC}{PQ}$ [supposition]; $\therefore \ \frac{AB}{DE} = \frac{BC}{EF}$	
$\therefore \ EF = PQ$	
Therefore, ΔAPQ and ΔDEF are congruent.	
$\therefore$ $\angle PAQ = \angle EDF, \angle APQ = \angle DEF . \angle AQP = \angle DFE$,	[SSS Theorem]
$\therefore \ \angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$	
$\angle A = \angle D, \ \angle B = \angle E, \ \angle C = \angle F$.	

Theorem 7

If one angle of a triangle is equal to an angle of the other and the sides adjacent to the equal angles are proportional, the triangles are similar.

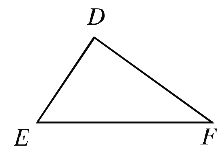
Proposition : Let in ΔABC and ΔDEF, $\angle A = \angle D$
and $\frac{AB}{DE} = \frac{AC}{DF}$.



It is to be proved that the triangles ΔABC and ΔDEF are similar.

Construction: Consider the matching sides of ΔABC and ΔDEF unequal. Take two points *P* and

Q on AB and AC respectively so that $AP = DE$ and $AQ = DF$. Join P and Q.



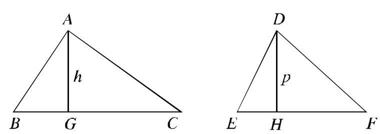
Proof:

Steps	Justification
(1) In ΔAPQ and ΔDEF, $AP = DE$, $AQ = DF$ and included $\angle A =$ included $\angle D$ $\therefore \Delta ABC \cong \Delta DEF$ $\therefore \angle A = \angle D, \angle APQ = \angle E, \angle AQP = \angle F$.	[SAS Theorem]
(2) Again, since $\frac{AB}{DE} = \frac{AC}{DF}$, so $\frac{AB}{AP} = \frac{AC}{AQ}$ $\therefore PQ \parallel BC$ Therefore, $\angle ABC = \angle APQ$ and $\angle ACB = \angle AQP$ $\therefore \angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ i.e., triangles ABC and DEF are equiangular. Therefore ΔABC and ΔDEF are similar.	[Theorem 2]

Theorem 8

The ratio of the areas of two similar triangles is equal to the ratio of squares on any two matching sides.

Proposition : Let the triangles ABC and DEF be similar and BC and EF be their matching sides respectively. It is required to prove that $\Delta ABC : \Delta DEF = BC^2 : EF^2$



Construction: Draw perpendiculars AG and DH on BC and EF respectively. Let $AG = h$, $DH = p$.

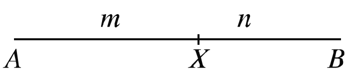
Proof:

Steps	Justification
(1) $\Delta ABC = \frac{1}{2} BC.h$ and $\Delta DEF = \frac{1}{2} EF.p$ $\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2} BC.h}{\frac{1}{2} EF.p} = \frac{h.BC}{p.EF} = \frac{h}{p} \times \frac{BC}{EF}$ (1) But in the triangles ABG and DEG, $\angle B = \angle E$, $\angle AGB = \angle DHE$ (= 1 right angle)	

$\therefore \angle BAG = \angle EDH$ (3) because ΔABC and ΔDEF are similar. $\frac{h}{p} = \frac{AB}{DE} = \frac{BC}{EF}$ $\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{h}{p} \times \frac{BC}{EF} = \frac{BC}{EF} \times \frac{BC}{EF} = \frac{BC^2}{EF^2}$	[Triangles ABC and DEF are similar]

14·1 Internal Division of a Line Segment in definite ratio

If A and B are two different points in a plane and m and n are two natural numbers, we acknowledge that there exits a unique point X lying between A and B and $AX : XA = m : n$.

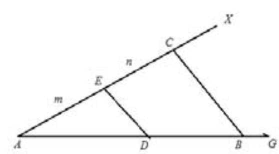


In the above figure, the line segment AB is divided at X internally in the ratio $m : n$, i.e. $AX : XB = m : n$.

Construction 1

To divide a given line segment internally in a given ratio.

Let the line segment AB be divided internally in the ration $m : n$.



Construction: Let an angle $\angle BAX$ be drawn at A. From AX cut the lengths $AE = m$ and $EC = n$ sequentially. Join B, C. At E, draw line segment ED parallel to CB which intersects AB at D. Then the line segment AB is divided at D internally in the ratio m : n.

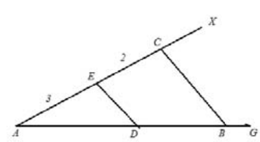
Proof: Since the line segment DE is parallel to a side BC of the triangle ABC

$\therefore AD : DB = AE : EC = m : n.$

Activity :
1. Divide a given line segment in definite ratio internally by an alternative method.

Example 1. Divide a line segment of length 7cm internally in the ratio 3.2.

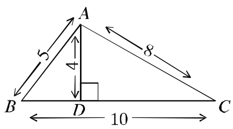
Solution: Draw any ray AG. From AG, cut a line segment AB =7cm. Draw an angle $\angle BAX$ at A.



From AX, cut the lengths $AE = 3$ cm and $EC = 2$ cm. from EX. Join B, C. At E, draw an angle $\angle AED$ equal to $\angle ACB$ whose side intersects AB at D. Then the line segment AB is divided at D internally in the ratio 3:2.

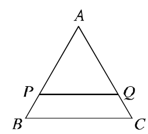
Exercise 14·2

1. Consider the following information:
   i. ratios are considered to compare two expressions
   ii. to find ratio, expressions are measured in the same unit
   iii. to find ratio, expressions must be of the same type.
   Which case of the following is true?
   a. i and ii b. ii and iii c. i and iii d. i, ii and iii

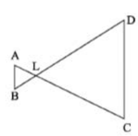


Use the information from the above figure to answer the questions 2 and 3.

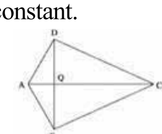
2. What is the ratio of the height and base of the triangle *ABC*?
   a. $\frac{1}{2}$ b. $\frac{4}{5}$ c. $\frac{2}{5}$ d. $\frac{5}{4}$
3. What is the area of triangle *ABD* in sq. units?
   a. 6 b. 20 c. 40 d. 60
4. In triangle *ABC*, if $PQ \parallel BC$, which of the following is true?
   a. $AP : PB = AQ : QC$ b. $AB : PQ = AC : PQ$
   c. $AB : AC = PQ : BC$ d. $PQ : BC = BP : BQ$



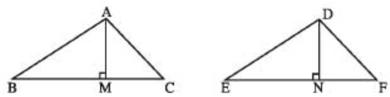
5 In a square how many lines of symmetry are there?
   a. 10 b. 8 c. 6 d. 4
6 Prove that if each of the two triangles is similar to a third triangle, they are congruent to each other.
7 Prove that, if one acute angle of a right angled triangle is equal to an acute angle of another right angled triangle, the triangles are similar.
8 Prove that the two right angled triangles formed by the perpendicular from the vertex containing the right angle are similar to each other and also to the original triangle.
9 In the adjacent figure, $\angle B = \angle D$ and $CD = 4AB$. Prove that $BD = 5BL$.



10. A line segment drawn through the vertex *A* of the parallelogram *ABCD* intersects the *BC* and *DC* at *M* and *N* respectively. Prove that $BM \times DN$ is a constant.
11. In the adjacent figure, $BD \perp AC$ and
$DQ = BA = 2AQ = \frac{1}{2}QC$. $BD = 5BL$. Prove that, $DA \perp DC$.



12. In the triangles ABC and DEF $\angle A = \angle D$. Prove that,
ΔABC t $\Delta DEF = AB.AC$ t $DE.DF$.
13. The bisector AD of $\angle A$ of the triangle ABC intersects BC at D. The line segment CE parallel to DA intersects the line segment BA extended.
a. Draw the specified figure.
b. Prove that ,$BD : DC = BA : AC$.
c. If a line segment parallel to BC intersect AB and AC at P and Q respectively, prove that $BD : DC = BP : CQ$.
14. In the figure, ABC and DEF are two similar triangles.
a. Name the matching sides and matching angles of the triangles.
b. Prove that,

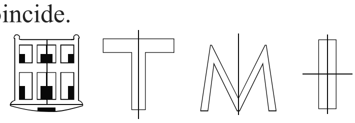


$$\frac{\Delta ABC}{\Delta DEF} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2} = \frac{BC^2}{EF^2}$$

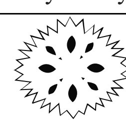
c. If $BC = 3$ cm , $EF = 8$ cm , $\angle B = 60^\circ$, $\frac{BC}{AB} = \frac{3}{2}$ and $\angle ABC = 3$ sq cm,
draw the triangle DEF and find its area.

14.4 Symmetry

Symmetry is an important geometrical concept, commonly exhibited in nature and is used almost in every field of our activity. Artists, designers, architects, carpenters always make use of the idea of symmetry. The tree-leaves, the flowers, the beehives, houses, tables, chairs -everywhere we find symmetrical designs. A figure has line symmetry, if there is a line about which the figure may be folded so that the two parts of the figure will coincide.

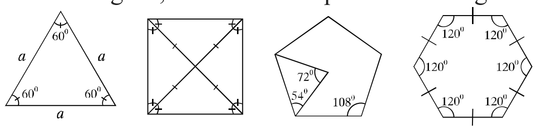


Each of the above figures has the line of symmetry. The last figure has two lines of symmetry

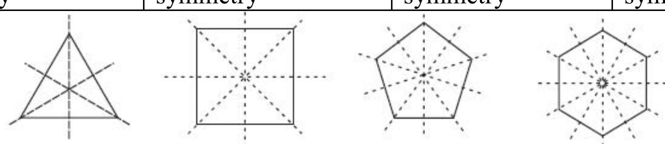
Activity: 1. Sumi has made some paper-cut design as shown in the adjacent figure. In the figure, mark the lines of symmetry. How many lines of symmetry does the figure have? 2. Write and identify the letters in English alphabet having line symmetry. Also mark their line of sysmmetry.	

14.5 Lines of Symmetry of Regular Polygons

A polygon is a closed figure made of several line segments. A polygon is said to be regular if all its sides are of equal length and all its angles are equal. The triangle is a polygon made up of the least number of line segments. An equilateral triangle is a regular polygon of three sides. An equilateral triangle is regular because its sides as well as angles are equal. A square is the regular polygon of four sides. The sides of a square are equal and each of the angles is equal to one right angle. Similarly, in regular pentagons and hexagons, the sides are equal and the angles are equal as well.



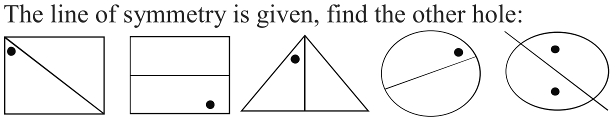
Each regular polygons is a figure of symmetry. Therefore, it is necessary to know their lines of symmetry. Each regular polygon has many lines of symmetry as it has many sides.

Three lines of symmetry	Four lines of symmetry	Five lines of symmetry	Six lines of symmetry
			
Equilateral triangle	Square	Regular pentagon	Regular hexagon

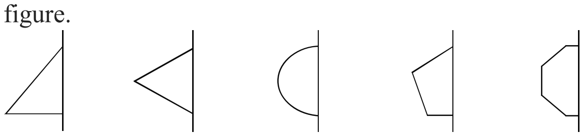
The concept of line symmetry is closely related to mirror reflection. A geometrical figure, has line symmetry when one half of it is the mirror image of the other half. So, the line of symmetry is also called the reflection symmetry.

Exercise 14.3

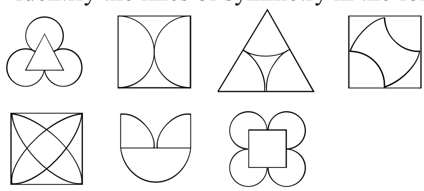
1. Which of the following figures have line symmetry?
   (a) A house (b) A mosque (c) A temple (d) A church (e) A pagoda
   (f) Parliament house (g) The Tajmahal.
2. The line of symmetry is given, find the other hole:



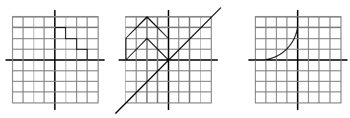
3. In the following figures, the line of symmetry is given; complete and identify the figure.



4. Identify the lines of symmetry in the following geometrical figures:



5. Complete each of the following incomplete geometrical shapes to be symmetric about the mirror line:



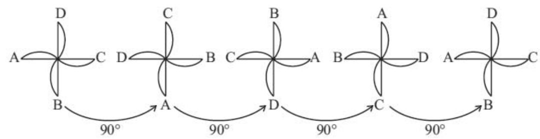
6. Find the number of lines of symmetry of the following geometrical figures:
   (a) An isosceles triangle (b) A scalene triangle (c) A square
   (d) A rhombus (e) A pentagon (f) A regular hexagon
   (g) A circle
7. Draw the letters of the English alphabet which have reflection symmetry with respect to
   (a) a vertical mirror
   (b) a horizontal mirror
   (c) both horizontal and vertical mirrors.
8. Draw three examples of shapes with no line of symmetry.

14.6 Rotational Symmetry

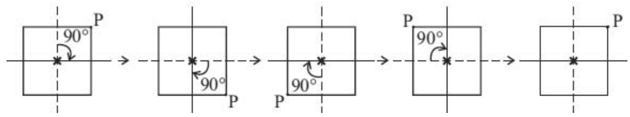
When an object rotates around any fixed point, its shape and size do not change. But the different parts of the object change their position. If the new position of the object after rotation becomes identical to the original position, we say the object has a **rotational symmetry.** The wheels of a bicycle, ceiling fan, square are examples of objects having rotational symmetry etc.. As a result of rotation the blades of the fan looks exactly the same as the original position more than once. The blades of a fan may rotate in the clockwise direction or in the anticlockwise direction. The wheels of a bicycle may rotate in the clockwise direction or in the anticlockwise direction. The rotation in the anti clockwise direction is considered the positive direction of rotation.

This fixed point around which the object rotates is the **centre of rotation.** The angle of turning during rotation is called the **angle of rotation**. A full-turn means rotation by 360°; a half-turn is rotation by 180°.

In the figure below, a fan with four blades rotating by 90° is shown in different positions. It is noted that us a fall turn of the four positions (rotating about the angle by 90°, 180°, 270° and 360°), the fan looks exactly the same. For this reason, it is said that the rotational symmetry of the fan is order 4.



Here is one more example for rotational symmetry. Consider the intersection of two diagonals of a square the centre of rotation. In the quarter turn about the centre of the square, any diagonal position will be as like as the second figure. In this way, when you complete four quarter-turns, the square re aches its original position. It is said that a square has a **rotational symmetry of order 4.**

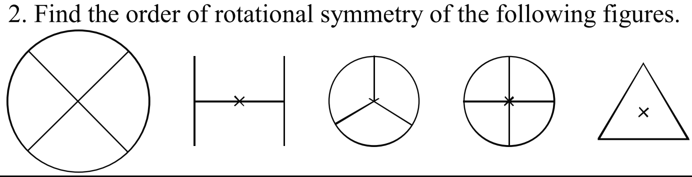


Observe also that every object occupies sa me position after one complete revolution. So every geometrical object has a rotationa l symmetry of order 1. Such cases have no interest for us. For finding the rotational symmetry of an object, one need to consider the following matter.

(a) The centre of rotation
(b) The angle of rotation
(c) The direction of rotation
(d) The order of rotational symmetry.

Activity:

1. Give examples of 5 plane objects from your surroundings which have rotational symmetry.
2. Find the order of rotational symmetry of the following figures.



14·7 Line Symmetry and Rotational Symmetry

We have seen that some geometrical shapes have only line symmetry, some have only rotational symmetry and some have both line symmetry and rotational symmetry. For example, the square has four lines of symmetry as well rotational symmetry of order 4.

The circle is the most symmetrical figure, because it can be rotated around its centre through any angle. Therefore, it has unlimited order of rotational of symmetry. At the same time, every line through the centre forms a line of reflection symmetry and so it has unlimited number of lines of symmetry.

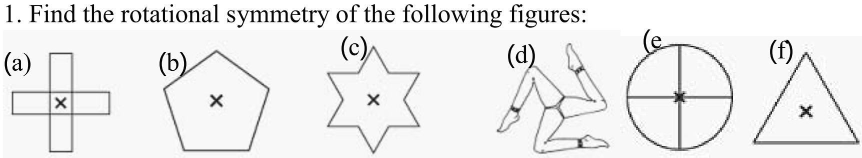
Activity:

1. Determine the line of symmetry and the rotational symmetry of the given alphabet and complete the table below:

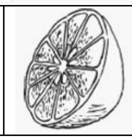
Letter	Line of symmetry	Number of lines of symmetry	Rotational symmetry	Order of rotational symmetry
Z	No	0	Yes	2
H				
O				
E				
C				

Exercise 14·4

1. Find the rotational symmetry of the following figures:



2. When you slice a lemon the crosssection looks as shown in the figure. Determine the rotational symmetry of the figure.



3 Fill in the blanks:

Shape	Centre of Rotation	Order of Rotation	Angle of Rotation
Square			
Rectangle			
Rhombus			
Equilateral triangle			
Semicircle			
Regular pentagon			

4. Name the quadrilaterals which have line of symmetry and rotational symmetry of order more than 1.

5. Can we have a rotational symmetry of a body of order more than 1 whose angle of rotation is 18 Justify your answer.

Chapter Fifteen

Area Related Theorems and Constructions

We know that bounded plane figures may have different shapes. If the region is bounded by four sides, it is known as quadrilateral. Quadrilaterals have classification and they are also named after their shapes and properties. Apart from these, there are regions bounded by more than four sides. These are polygonal regions or simply polygons. The measurement of a closed region in a plane is known as area of the region. For measurement of areas usually the area of a square with sides of 1 unit of length is used as the unit area and their areas are expressed in square units. For example, the area of Bangladesh is 1.4 lacs square kilometres (approximately). Thus, in our day to day life we need to know and measure areas of polygons for meeting the necessity of life. So, it is important for the learners to have a comprehensive knowledge about areas of polygons. Areas of polygons and related theorems and constructions are presented here.

At the end of the chapter, the students will be able to

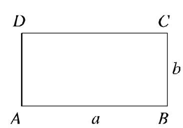
- Explain the area of polygons
- Verify and prove theorems related to areas
- Construct polygons and justify construction by using given data
- Construct a quadrilateral with area equal to the area of a triangle
- Construct a triangle with area equal to the area of a quadrilateral

15·1 Area of a Plane Region

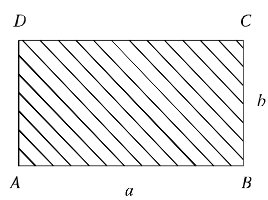
Every closed plane region has definite area. In order to measure such area, usually the area of a square having sides of unit length is taken as the unit. For example, the area of a square with a side of length 1 cm. is 1 square centimetre.

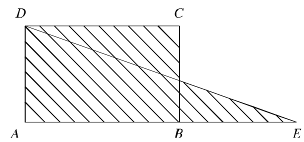
We know that,

(a) In the rectangular region *ABCD* if the length $AB = a$ units (say, metre), breadth $BC = b$ units (say, metre), the area of the region $ABCD = ab$ square units (say, square metres).

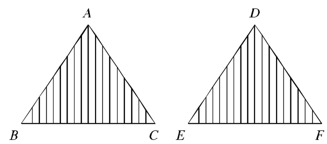


(b) In the square region *ABCD* if the length of a side $AB = a$ units (say, metre), the area of the region $ABCD = a^2$ square units (say, square metres).



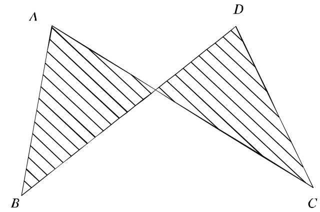


When the area of two regions are equal, the sign '=' is used between them. For example, in the figure the area of the rectangular region *ABCD* = Area of the triangular region *AED*.



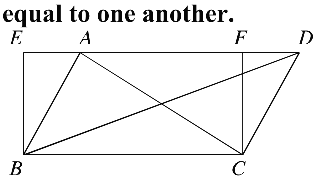
It is noted that if ΔABC and ΔDEF are congruent, we write $\Delta ABC \cong \Delta DEF$. In this case, the area of the triangular region *ABC* = area of the triangular region *DEF*.

But, two triangles are not necessarily congruent when they have equal areas. For example, in the figure, area of ΔABC = area of ΔDBC but ΔABC and ΔDBC are not congruent.



Theorem 1

Areas of all the triangular regions having same base and lying between the same pair of parallel lines are equal to one another.



Let the triangular regions *ABC* and *DBC* stand on the same base *BC* and lie between the pair of parallel lines *BC* and *AD*. It is required to prove that, Δ region *ABC* = Δ region *DBC*.

Construction : At the points *B* and *C* of the line segment *BC,* draw perpendiculars *BE* and *CF* respectively. They intersect the line *AD* or *AD* produced at the points *E* and *F* respectively. As a result a rectangular region *EBCF* is formed.

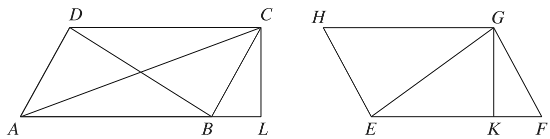
Proof : According to the construction, *EBCF* is a rectangular region. The triangular region *ABC* and rectangular region *EBCF* stand on the same base *BC* and lie between the two parallel line segments *BC* and *ED*. Hence, Δ region *ABC* = $\frac{1}{2}$ (rectangular region *EBCF*)

Similarly, Δ region *DBC* = $\frac{1}{2}$ (rectangular region *EBCF*)

∴ Δ region *ABC* = Δ region *DBC* (proved).

Theorem 2

Parallelograms lying on the same base and between the same pair of parallel lines are of equal area.



Let the parallelograms regions *ABCD* and *EFGH* stand on the same base and lie between the pair of parallel lines *AF* and *DG* and *AB* = *EF*. It is required to prove that, area of the parallelogram *ABCD* = area of the parallelogram *EFGH*.

Construction :
The base *EF* of *EFGH* is equal. Join *AC* and *EG*. From the points *C* and *G,* draw perpendiculars *CL* and *GK* to the base *AF* respectively.

Proof: The area of $\Delta ABC = \frac{1}{2} AB \times CL$ and the area of ΔEFG is $\frac{1}{2} EF \times GK$.

$\because$ $AB = EF$ and $CL = GK$ (by construction)

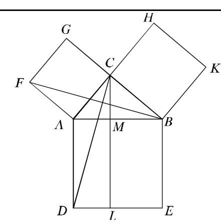
Therefore, area of ΔABC = area of the triangle *EFG*

$\Rightarrow \frac{1}{2}$ area of the parallelogram *ABCD* = $\frac{1}{2}$ area of the parallelogram *EFGH*

Area of the parallelogram *ABCD* = area of the parallelogram *EFGH*. (Proved)

Theorem 3 (Pythagoras Theorem)

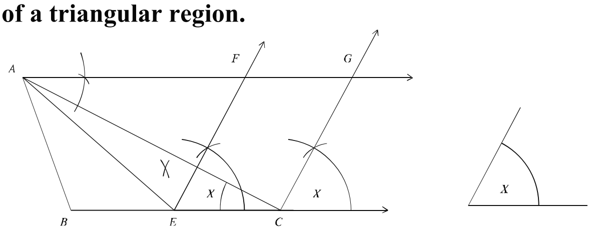
In a right angles triangle, the square of the hypotenuse is equal to the sum of squares of other two sides.

Proposition: Let *ABC* be a right angled triangle in which $\angle ACB$ is a right angle and hypotenuse is *AB*. It is to be proved that $AB^2 = BC^2 + AC^2$. **Construction:** Draw three squares *ABED*, *ACGF* and *BCHK* on the external sides of *AB*, *AC* and *BC* respectively. Through *C,* draw the line segment *CL* parallel to *AD* which intersects *AB* and *DE* at *M* and L respectively. Join *C*, *D* and *B*, *F*. **Proof:**	
Steps	Justification
(1) In $\Delta\ CAD$ and $\Delta\ FAB$, $CA = AF$, $AD = AB$ and included $\angle CAD = \angle CAB + \angle BAD$ $= \angle CAB + \angle CAF$ = included $\angle BAF$	[$\angle BAD = \angle CAF$ = 1 right angle]

Therefore, $\Delta\, CAD \cong \Delta\, FAB$	
(2) Triangular region CAD and rectangular region $ADLM$ lie on the same base AD and between the parallel lines AD and CL .	[SAS theorem]
Therefore, Rectangular region $ADLM$ = 2(triangular region CAD)	[Theorem 1]
(3) Triangular region BAF and the square $ACGF$ lie on the same base AF and between the parallel lines AF and BG .	
Hence Square region $ACGF$ = 2(triangular region FAB) = 2(triangular region CAD)	[Theorem 1]
(4) Rectangular region $ADLM$ = square region $ACGF$	[From (2) and (3)]
(5) Similarly joining C, E and A, K , it can be proved that rectangular region $BELM$ = square region $BCHK$	
(6) Rectangular region ($ADLM + BELM$)= square region $ACGF$ + square region $BCHK$	[From (4) and (5)]
or, square region $ABED$ = square region $ACGF$ + square region $BCHK$	
That is, $AB^2 = BC^2 + AC^2$ [Proved]	

Construction 1

Construct a parallelogram with an angle equal to a definite angle and area equal to that of a triangular region.



Let ABC be a triangular region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle equal to $\angle x$ and area equal to the area of the triangular region ABC.

Construction:

Bisect the line segment BC at E. At the point E of the line segment EC, construct $\angle CEF$ equal to $\angle x$. Through A, construct AG parallel to BC which intersects the ray EF at F. Again, through C, construct the ray CG parallel to EF which intersects the ray AG at G. Hence, $ECGF$ is the required parallelogram.

Proof: Join A, E. Now, area of the triangular region ABE = area of the triangular region AEC [since base BE = base EC and heights of both the triangles are equal]

$\therefore$ area of the triangular region ABC = 2 (area of the triangular region AEC).

Again, area of the parallelogram region *ECGF* is 2 (area of the triangular region *AEC*) [since both lie on the same base and $EC \parallel AG$).

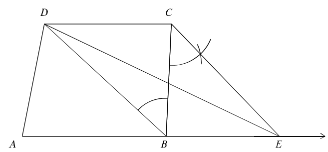
$\therefore$ area of the parallelogram region *ECGF* = area of the triangular region *ABC*.

Again $\angle CEF = \angle x$ [since $EF \parallel CB$ by construction]

S the parallelogram EG is the required parallelogram.

Construction 2

Construct a triangle with area of the triangular region equal to that of a quadrilateral region.



Let *ABCD* be a quadrilateral region. To construct a triangle such that area of the is triangular region equal to that of a rectangular region *ABCD*.

Construction :

Join *D*, *B*. Through *C,* construct *CE* parallel to *DB* which intersects the side *AB* extended at *E*. Join *D, E*. Then, ΔDAE is the required triangle.

Proof: The triangles *BDC* and *BDE* lie on the same base *BD* and $DB \parallel CE$ (by construction).

$\therefore$ area of the triangular region *BDC* = area of the triangular region *BDE*.

$\therefore$ area of the triangular region *BDC* + area of the triangular region *ABD* = area of the triangular region *BDE* + area of the triangular region *ABD*

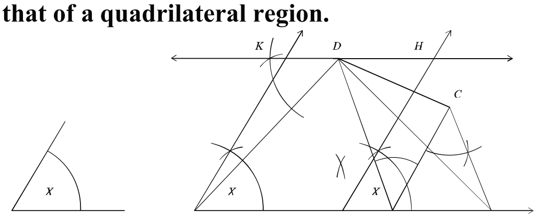
$\therefore$ area of the quadrilateral region *ABCD* = area of the triangular region *ADE* .

Therefore, ΔADE is the required triangle.

N.B. Applying the above mentioned method innumerable numbers of triangles can be drawn whose area is equal to the area of a given quadrilateral region.

Construction 3

Construct a parallelogram, with a given an angle and the area of the bounded region equal to that of a quadrilateral region.



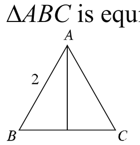
Let *ABCD* be a quadrilateral region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle $\angle x$ and the area equal to area of the quadrilateral region *ABCD*.

Construction:
Join B, D. Through C construct CF parallel to DB which intersects the side AB extended at F. Find the midpoint G of the line segment AF. At A of the line segment AG, construct $\angle GAK$ equal to $\angle x$ and draw $GH \parallel AK$ through G. Again draw $KDH \parallel AG$ through D which intersects AK and GH at K and H respectively.
Hence $AGHK$ is the required parallelogram.
Proof: Join D, E. By construction $AGHK$ is a parallelogram. where $\angle GAK = \angle x$. Again, area of the triangular region DAF = area of the rectangular region $ABCD$ and area of the parallelogram $AGHK$ = area of the triangular region DAF. Therefore, $AGHK$ is the required parallelogram.

Exercise 15

1. The lengths of three sides of a triangle are given, in which case below the construction of the right angled triangle is not possible ?
   (a) 3 cm, 4 cm, 5 cm (b) 6 cm, 8 cm, 10 cm
   (c) 5 cm, 7 cm, 9 cm (d) 5 cm, 12 cm, 13 cm
2. Observe the following information :
   i. Each of the bounded plane has definite area.
   ii. If the area of two triangles is equal, the two angles are congruent.
   iii. If the two angles are congruent, their area is equal.

Which one of the following is correct ?
   (a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii
   Answer question no. 3 and 4 on the basis of the information in the figure below, ΔABC is equilateral, $AD \perp BC$ and $AB = 2$:



3. BD = What ?
   (a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4
4. What is the height of the triangle ?
   (a) $\frac{4}{\sqrt{3}}$ sq. unit (b) $\sqrt{3}$ sq. unit (c) $\frac{2}{\sqrt{3}}$ sq. unit (d) $2\sqrt{3}$ sq. unit.
5. Prove that the diagonals of a parallelo gram divide the parallelogram into four equal triangular regions.
6. Prove that the area of a square is half the area of the square drawn on its diagonal.

7. Prove that any median of a triangle divides the triangular region into two regions of equal area.
8. A parallelogram and a rectangular region of equal area lie on the same side of the bases. Show that the perimeter of the parallelogram is greater than that of the rectangle.
9. *X* and *Y* are the mid points of the sides *AB* and *AC* of the triangle *ABC*. Prove that the area of the triangular region $AXY = \frac{1}{4}$ (area of the triangular region *ABC*)
10. In the figure, *ABCD* is a trapezium with sides *AB* and *CD* parallel. Find the area of the region bounded by the trapezium *ABCD*.
11. *P* is any point interior to the parallelogram *ABCD*. Prove that the area of the triangular region *PAB* + the area of the triangular region $PCD = \frac{1}{2}$ (area of the parallelogram *ABCD*).
12. A line parallel to *BC* of the triangle *ABC* intersects *AB* and *AC* at *D* and *E* respectively. Prove that the area of the triangular region *DBC* = area of the triangular region *EBC* and area of the triangular region *DBF* = area of the triangular region *CDE*.
13. $\angle A = 1$ right angle of the triangle *ABC*. *D* is a point on *AC*. Prove that $BC^2 + AD^2 = BD^2 + AC^2$.
14. *ABC* is an equilateral triangle and *AD* is perpendicular to *BC*. Prove that $4AD^2 + 3AB^2$.
15. *ABC* is an isosceles triangle. *BC* is its hypotenuse and *P* is any point on *BC*. Prove that $PB^2 + PC^2 = 2PA^2$.
16. *C* is an obtuse angle of ΔABC; *AD* is a perpendicular to *BC*. Show that. $AB^2 = AC^2 + BC^2 + 2BC.CD$
17. *C* is an acute angle of ΔABC ; *AD* is a perpendicular to *BC*. Show that $AB^2 = AC^2 + BC^2 - 2BC.CD$.
18. *AD* is a median of ΔABC. Show that, $AB^2 + AC^2 = 2(BD^2 + AD^2)$.

Chapter Sixteen

Mensuration

The length of a line, the area of a place, the volume of a solid etc. are determined for practical purposes. In the case of measuring any such quantity, another quantity of the same kind having some definite magnitude is taken as unit. The ratio of the quantity measured and the unit defined in the above process is the amount of the quantity.

i.e. magnitude $= \dfrac{\text{Quantity measured}}{\text{Unit quantity}}$

In the case of a fixed unit, every magnitude is a number which denotes how many times of the unit of the magnitude is the magnitude of the quantity measured. For example, the bench is 5 metre long. Here metre is a definite length which is taken as a unit and in comparison to that the bench is 5 times in length.

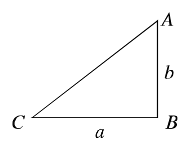
At the end of the Chapter, the students will be able to –

- Determine the area of polygonal region by applying the laws of area of triangle and quadrilateral and solve allied problems.
- Determine the circumference of the circle and a length of the chord of a circle.
- Determine the area of circle.
- Determie the area of circle. Determining the area of a circle and its part, solve the alied problems.
- Determine the area of solid rectangles, cubes and cylinder and solve the allied problems.
- Determine the area of uniform and non uniform polygonal regions.

16·1 Area of Triangular region

In the previous class, we learned that area of triagular region = $\frac{1}{2}$ base × height.

(1) **Right Angled Triangle :**

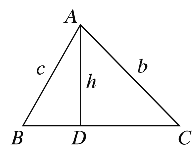


Let in the right angled triangle ABC, $BC = a$ and $AB = b$ are the adjacent sides of the right angle. Here if we consider BC the base and AB the height,

$$\text{Area of } \Delta ABC = \frac{1}{2} \text{ base} \times \text{height}$$
$$= \frac{1}{2}ab$$

(2) Two sides of a triangular region and the angle included between them are given.

Let in ΔABC the sides are :
$BC = a$, $CA = b$, $AB = c$.
AD is drawn perpendicular from A to BC.
Let altitude (height) $AD = h$.



Considering the angle C, we get, $\frac{AD}{CA} = \sin C$

or, $\frac{h}{b} = \sin C$ or, $h = b \sin C$

Area of $\Delta\ ABC = \frac{1}{2} BC \times AD$

$= \frac{1}{2} a \times b \sin C$

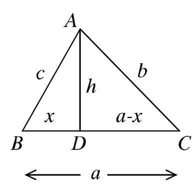
$= \frac{1}{2} ab \sin C$

Similarly, area of $\Delta\ ABC = \frac{1}{2} bc \sin A$

$= \frac{1}{2} ca \sin B$

(3) Three sides of a triangle are given.

Let in ΔABC, $BC = a$, $CA = b$ and $AB = c$.
$\therefore$ Perimeter of the triangle $2s = a + b + c$
We draw $AD \perp BC$
Let, $BD = x$, so $CD = a - x$
In right angled ΔABD and ΔACD



$\therefore AD^2 = AB^2 - BD^2$ and $AD^2 = AC^2 - CD^2$
$\therefore\ AB^2 - BD^2 = AC^2 - CD^2$
or, $c^2 - x^2 = b^2 - (a - x)^2$
or, $c^2 - x^2 = b^2 - a^2 + 2ax - x^2$
or, $2ax = c^2 + a^2 - b^2$

$\therefore\ x = \frac{c^2 + a^2 - b^2}{2a}$

Again, $AD^2 = c^2 - x^2$

$= c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$

$= \left(c + \frac{c^2 + a^2 - b^2}{2a}\right)\left(c - \frac{c^2 + a^2 - b^2}{2a}\right)$

$$= \frac{2ac + c^2 + a^2 - b^2}{2a} \cdot \frac{2ac - c^2 - a^2 + b^2}{2a}$$

$$= \frac{\{(c+a)^2 - b^2\}\{b^2 - (c-a)^2\}}{4a^2}$$

$$= \frac{(a+b+c)(a+b+c-2b)(a+b+c-2a)(a+b+c-2c)}{4a^2}$$

$$= \frac{2s(2s-2b)(2s-2a)(2s-2c)}{4a^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$\therefore\ AD = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

Area of $\Delta\ ABC = \frac{1}{2} BC \cdot AD$

$$= \frac{1}{2} \cdot a \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(b-c)}$$

$$= \sqrt{s(s-a)(s-b)(b-c)}$$

(4) Equilateral Triangle :

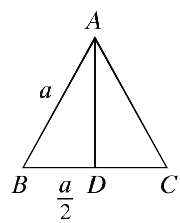
Let the length of each side of the equilateral triangular region ABC be a.

Draw $AD \perp BC \ \therefore\ BD = CD = \frac{a}{2}$

In right angled ΔABD

$$BD^2 + AD^2 = AB^2$$

or, $AD^2 = AB^2 - BD^2 = a^2 - \left(\frac{a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

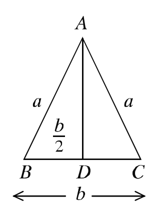


$$\therefore\ AD = \frac{\sqrt{3}a}{2}$$

Area of $\Delta\ ABC = \frac{1}{2} \cdot BC \cdot AD$

$$= \frac{1}{2} \cdot a \cdot \frac{\sqrt{3}a}{2} \text{ or, } \frac{\sqrt{3}}{4}a^2$$

(5) Isosceles triangle :



Let ABC be an isosceles triangle in which $AB = AC = a$ and $BC = b$

Draw $AD \perp BC$. $\therefore$ $BD = CD = \frac{b}{2}$

In ΔABD right angled

$\therefore$ $AD^2 = AB^2 - BD^2 = a^2 - \left(\frac{b}{2}\right)^2 = a^2 - \frac{b^2}{4} = \frac{4a^2 - b^2}{4}$

$\therefore$ $AD = \frac{\sqrt{4a^2 - b^2}}{2}$

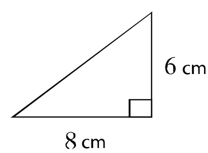
Area of isosceles Δ ABC $= \frac{1}{2} \cdot BC \cdot AD$

$= \frac{1}{2} \cdot b \cdot \frac{\sqrt{4a^2 - b^2}}{2}$

$= \frac{b}{4}\sqrt{4a^2 - b^2}$

Example 1. The lengths of the two sides of a right angled triangle, adjacent to right angle are 6 cm. and 8 cm. respectively. Find the area of the triangle.

Solution : Let, the sides adjacent to right angle are $a = 8$ cm. and $b = 6$ cm. respectively.



$\therefore$ Its area = $\frac{1}{2}ab$

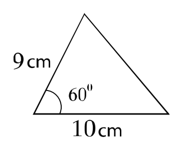
$= \frac{1}{2} \times 8 \times 6$ square cm. = 24 square cm.

Required area 24 square cm.

Example 2. The lengths of the two sides of a triangle are 9 cm. and 10 cm. respectively and the angle included between them is 60°. Find the area.

Solution : Let, the sides of triangle are $a = 9$ cm. and $b = 10$ cm. respectively.

Their included angle $\theta = 60^\circ$.



$\therefore$ Area of the triangle= $\frac{1}{2}ab\sin 60^o$

$= \frac{1}{2} \times 9 \times 10 \times \frac{\sqrt{3}}{2}$ sq. cm.

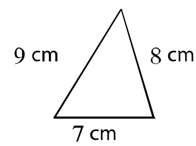
$= 38 \cdot 97$ sq. cm. (approx)

Required area $38 \cdot 97$ sq. cm. (approx)

Exmple 3. The lengths of the three sides of a triangle are 7 cm., 8 cm. and 9 cm. respectively. Find its area.

Solution : Let, the lengths of the sides of the triangle are $a = 7$ cm., $b = 8$ cm. and $c = 9$ cm.

$\therefore$ Semi perimeter $s = \dfrac{a+b+c}{2} = \dfrac{7+8+9}{2}$ cm. = 12 cm.



$\therefore$ Its area $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{12(12-7)(12-8)(12-9)}$ sq. cm.

$= \sqrt{12 \times 5 \times 6 \times 7}$ sq. cm. $= 50 \cdot 2$ sq. cm. (approx)

$\therefore$ The area of the triangle is $50 \cdot 2$ sq. cm. (approx)

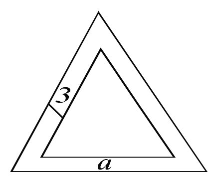
Example 4. The area of an equilateral triangle increases by $3\sqrt{3}$ sq. metre when the length of each side increases by 1 metre. Find the length of the side of the triangle.

Solution : Let, the length of each side of the equilateral triangle is a metre.

$\therefore$ Its area $= \dfrac{\sqrt{3}}{4} a^2$ sq. m.

The area of the triangle when the length of each side increases by 1m. $= \dfrac{\sqrt{3}}{4}(a+1)^2$ sq. metre.

According to the question, $\dfrac{\sqrt{3}}{4}(a+1)^2 - \dfrac{\sqrt{3}}{4}a^2 = 3\sqrt{3}$



or, $(a+1)^2 - a^2 = 12$; [divided by $\dfrac{\sqrt{3}}{4}$]

or, $a^2 + 2a + 1 - a^2 = 12$ or, $2a = 11$ or, $a = 5 \cdot 5$

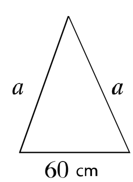
The required length is $5 \cdot 5$ metre.

Example 5. The length of the base of an isosceles triangle is 60 cm. If its area is 1200 sq. metre, find the length of equal sides.

Solution : Let the base of the isosceles triangle be $b = 60$ cm. and the length of equal sides be a.

$\therefore$ Area of the triangle $= \dfrac{b}{4}\sqrt{4a^2 - b^2}$

According to the question, $\dfrac{b}{4}\sqrt{4a^2 - b^2} = 1200$



or, $\dfrac{60}{4}\sqrt{4a^2 - (60)^2} = 1200$

or, $15\sqrt{4a^2 - 3600} = 1200$

or, $\sqrt{4a^2 - 3600} = 80$

or, $4a^2 - 3600 = 6400$; [by squaring]
or, $4a^2 = 10000$
or, $a^2 = 2500$
$\therefore\ a = 50$

$\therefore$ The length of equal sides of the triangle is 50 cm.

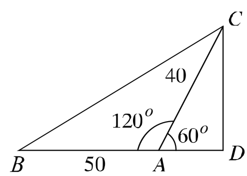
Example 6. From a definite place two roads run in two directions making are angle 120^o. From that definite place, persons move in the two directions with speed of 10 km per hour and 8 km per hour respectively. What will be the direct distance between them after 5 hours ?

Solution : Let two men start from A with velocities 10 km/hour and 8 km/hour respectively and reach B and C after 5 hours. Then after 5 hours, the direct distance between them is BC. From C perpendicular CD is drawn on BA produced.

$\therefore\ AB = 5 \times 10$ km = 50 km, $AC = 5 \times 8$ km = 40 km.

and $\angle BAC = 120^o$

$\therefore\ \angle DAC = 180° - 120° = 60°$



From the right angled triangle ACD

$\therefore\ \dfrac{CD}{AC} = \sin 60^o$ or, $CD = AC \sin 60^o = 40 \times \dfrac{\sqrt{3}}{2} = 20\sqrt{3}$

and $\dfrac{AD}{AC} = \cos 60^o$ or, $AD = AC \cos 60^o = 40 \times \dfrac{1}{2} = 20$

Again, we get from right angled ΔBCD,

$BC^2 = BD^2 + CD^2 = (BA + AD)^2 + CD^2$
$= (50 + 20)^2 + (20\sqrt{3})^2 = 4900 + 1200 = 6100$

$\therefore\ BC = 78 \cdot 1$ (app.)

Required distance is $78 \cdot 1$ km. (approx)

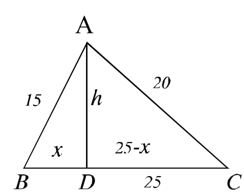
Example 7. The lengths of the sides of a triangle are 25, 20, 15 units respectively. Find the areas of the triangles in which it is divided by the perpendicular drawn from the vertex opposite of the greatest side.

Solution : Let in triangle ABC, $BC = 25$ units, $AC = 20$ units, $AB = 15$ units.

The drawn perpendicular AD from vertex A on side BC divides the triangular region into ΔABD and ΔACD.

Let $BD = x$ and $AD = h$

$\therefore\ CD = BC - BD = 25 - x$



In right angle ΔABD

$BD^2 + AD^2 = AB^2$ or, $x^2 + h^2 = (15)^2$

$\therefore\ x^2 + h^2 = 225$..........$(i)$

and ΔACD is right angled

$CD^2 + AD^2 = AC^2$ or, $(25 - x)^2 + h^2 = (20)^2$

or, $625 - 50x + x^2 + h^2 = 400$
or, $625 - 50x + 225 = 400$; [with the help of equation (i)]
or, $50x = 450$; $\therefore\ x = 9$
Putting the value of x in equation (i), we get,
$81 + h^2 = 225$ or, $h^2 = 144$ $\therefore\ h = 12$
Area of $\Delta\ ABD = \frac{1}{2} BD \cdot AD = \frac{1}{2} \times 9 \times 12$ square units $= 36$ square units
and area of $\Delta\ ACD = \frac{1}{2} BD \cdot AD = \frac{1}{2}(25 - 9) \times 12$ square units
$= \frac{1}{2} \times 16 \times 12$ square units $= 96$ square units
Required area is 36 square units and 96 square units.

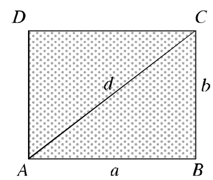
Exercise 16·1

1. The hypotenuse of a right angled triangle is 25 m. If one of its sides is $\frac{3}{4}$th of the other, find the length of the two sides.
2. A ladder with length 20m. stands vertically against a wall. How much further should the lower end of the end of the ladder be moved so that its upper end descends 4 metre?
3. The perimeter of an isosceles triangle is 16 m. If the length of equal sides is $\frac{5}{6}$th of base, find the area of the triangle.
4. The lengths of the two sides of a triangle are 25 cm., 27 cm. and perimeter is 84 cm. Find the area of the triangle.
5. When the length of each side of an equilateral triangle is increased by 2 metre, its area is increased by $6\sqrt{3}$ square metre. Find the length of side of the triangle.
6. The lengths of the two sides of a triangle are 26 m., 28 m. respectively and its area is 182 square metre. Find the angle between the two sides.
7. The perpendicular of a right angled triangle is 6cm less than $\frac{11}{12}$ times of the base, and the hypotenuse is 3 cm less than $\frac{4}{3}$ times of the base. Find the length of the base of the triangle.
8. The length of equal sides of an isosceles triangle is 10m and area 48 square metre. Find the length of the base.

9. Two roads run from a definite place with an angle of 135^o in two dinrections. Two persons move from the definite place in two directions with the speed of 7 km per hour and 5 km per hour respectively. What will be the direct distance between them after 4 hours?
10. If the lengths of the perpendiculars from a point interior of an equilateral triangle to three sides are 6 cm., 7 cm., 8 cm. respectively;find the length of sides of the triangle and the area of the triangular region.

16·2 Area of Quadrilateral Region

(1) Area of rectangular region



Let, the length of $AB = a$, breadth $BC = b$ and diagonal $AC = d$, of rectangle $ABCD$. We know, the diagonal of a rectangle divides. the rectangle into two equal triangular regions.

$\therefore$ Area of the rectangle $ABCD = 2\times$ area of $\Delta ABC = 2\times\frac{1}{2}a\cdot b = ab$

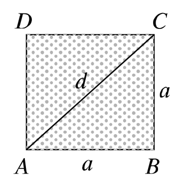
perimeter of the rectangular region, $s = 2(a+b)$

the ΔABC is right angled.

$AC^2 = AB^2 + BC^2$ or, $d^2 = a^2 + b^2$; $\therefore$ $d = \sqrt{a^2+b^2}$

(1) Area of square region

Let the length of each side of a square $ABCD$ be a and diagonal d. The diagonal AC divides the square region into two equal triangular regions.



$\therefore$ Area of square region $ABCD = 2\times$ area of $\Delta\ ABC$

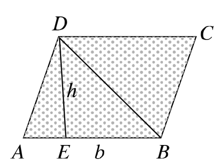
$$= 2\times\frac{1}{2}a\cdot a = a^2$$

Observe that, the perimeter of the square region $s = 4a$

and diagonal $d = \sqrt{a^2+a^2} = \sqrt{2a^2} = \sqrt{2}a$

(3) Area of parallelogram region.

(a) Base and height are given.



Let, the base $AB = b$ and height $DE = h$ of parallelogram $ABCD$. The diagonal BD divides the parallelogram into two equal triangular regions.

$\therefore$ The area of parallelogram $ABCD = 2\times$ area of $\Delta\ ABD$

$$= 2\times\frac{1}{2}b\cdot h$$

$$= bh$$

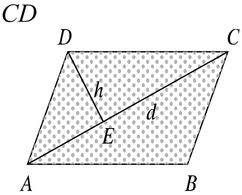
(b) The length of a diagonal and the length of a perpendicular drawn from the opposite angular point on that diagonal are given.

Let, the diagonal be $AC = d$ and the perpendicular from opposite angular point D on AC be $DE = h$ of a parallelogram $ABCD$. Diagonal AC divides the parallelogram into two equal triangular regions.

$\therefore$ The area of parallelogram region $ABCD$ = $2\times$ area of $\Delta\, ACD$

$$= 2\times\frac{1}{2}d\cdot h$$

$$= dh$$



(4) Area of Rhombus Region

Two diagonals of a rhombus region are given

Let the diagonals be $AC = d_1$, $BD = d_2$ of the rhombus $ABCD$ and the diagonals intersect each other at O. Diagonal AC divides the rhombus region into two equal triangular regions.

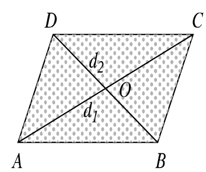
We know that the diagonals of a rhombus bisect each other at right angles.

$\therefore$ Height of $\Delta ACD = \frac{d_2}{2}$

$\therefore$ Area of the rhombus region $ABCD$ = $2\times$ area of $\Delta\, ACD$

$$= 2\times\frac{1}{2}d_1\times\frac{d_2}{2}$$

$$= \frac{1}{2}d_1 d_2$$



(5) Area of trapezium region

Two parallel sides of trapezium region and the distance of perpendicular between them are given.

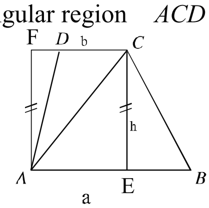
Let $ABCD$ be a trapezium whose lengths of parallel sides are $AB = a$ unit, $CD = b$ unit and distance between them be $CE = h$. Diagonal AC divides the trapezium region $ABCD$ into ΔABC and ΔACD.

Area of trapezium region $ABCD$

= Area of the triangular region ABC + Area of the triangular region ACD.

$$= \frac{1}{2}AB\times CE + \frac{1}{2}CD\times AF$$

$$= \left(\frac{1}{2}ah + \frac{1}{2}bh\right) = \frac{1}{2}h(a+b)$$



Example 1. Length of a rectangular room is $\frac{3}{2}$ times of breadth. If the area is 384 square metre, find the perimeter and length of the diagonal.

Solution : Let breath of the rectangular room is x metre

$\therefore$ Length of the room is $\frac{3x}{2}$ metre

$\therefore$ Area $\frac{3x}{2} \times x$ or, $\frac{3x^2}{2}$ square metre.

According to the question, $\frac{3x^2}{2} = 384$ or, $3x^2 = 768$ or, $x^2 = 256$; $\therefore$ $x = 16$ metre

$\therefore$ Length of the rectangular room = $\frac{3}{2} \times 16$ metre = 24 metre

and breath = 16 metre.

$\therefore$ Its perimeter = $2(24+16)$ metre = 80 metre.

and length of the diagonal = $\sqrt{(24)^2 + (16)^2}$ metre = $\sqrt{832}$ metre = $28 \cdot 84$ metre (app.)

The required perimetre is 80 metre and length of diagonal is $28 \cdot 84$ metre (approx)

Example 2. The area of a rectangular region is 2000 square metre. If the length is reduced by 10 metre, it becomes a square region. Find the length and breadth of the rectangular region.

Solution : Let length of the rectangular region be x metre and breadth y metre.

$\therefore$ Area of the rectangular region = xy square metre

According to the question $xy = 2000$................(1)

and $x - 10 = y$............(2)

We get, from equation (2) $y = x - 10$............(3)

From equations (1) and (3) we get,

$x(x-10) = 2000$ or, $x^2 - 10x - 2000 = 0$

or, $x^2 - 50x + 40x - 2000 = 0$ or, $(x-50)(x+40) = 0$

$\therefore$ $x - 50 = 0$ or, $x + 40 = 0$

or, $x = 50$ or, $x = -40$

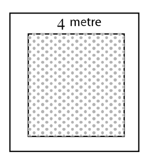
But length can never be negative.

$\therefore$ $x = 50$

Now puting the value of x in equation (3) we get

$y = 50 - 10 = 40$

$\therefore$ length = 50 m. and breadth = 40 m.



Example 3. There is a road of 4 metre width inside around a square field. If the area of the road is 1 hector, determine the area of the field excluding the road.

Solution : Let, the length of the square field is x metre.

$\therefore$ Its area is x^2 square metre.

There is a road around the field with width 4 m.

$\therefore$ Length of the square field excluding the road = $(x - 2 \times 4)$, or $(x - 8)$ m

$\therefore$ Area of the square field excluding the road is $(x-8)^2$ square m.

Area of the road = $\{x^2 - (x-8)^2\}$ square m.

Now, 1 hector = 10000 square m.

According the question, $x^2 - (x-8)^2 = 10000$

or, $x^2 - x^2 + 16x - 64 = 10000$

or, $16x = 10064$

$\therefore$ $x = 629$

Area of the square field excluding the road = $(629-8)^2$ square m.

= 385641 square m.

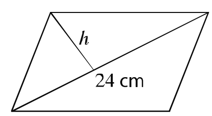
= $38 \cdot 56$ hector (approx.)

The required area is $38 \cdot 56$ hector (approx.)

Example 4. The area of a parallelogram is 120 sq. cm. and length of one of its diagonal is 24 cm. Determine the length of the perpendicular drawn on that diagonal from the opposite vertex.

Solution : Let a diagonal of a parallelogram be $d = 24$ cm. and the length of the perpendicular drawn on the diagonal from the opposite vertex be h cm.

$\therefore$ Area of the parallelogram = dh square cm.



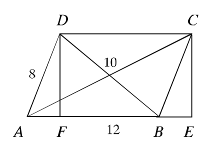
As per question, $dh = 120$ or, $h = \dfrac{120}{d} = \dfrac{120}{24} = 5$

The required length of the perpendicular is 5 cm.

Example 5. If the length of the sides of a parallelogram are 12 m. 8 m. If the length of the smaller diagonal is 10 m, determine the length of the other diagonal.

Solution : Let, in the parallelogram $ABCD$; $AB = a = 12$ m. and $AD = c = 8$ m. and diagonal $BD = b = 10$ m. Let us draw the perpendiculars DF and CE from D and C on the extended part of AB, respectively. Join A, C and B, D.

$\therefore$ Semi perrimeter of ΔABD is $s = \dfrac{12+10+8}{2}$ m. = 15 m.



$\therefore$ Area of the triangular region ABD = $\sqrt{s(s-a)(s-b)(s-c)}$

= $\sqrt{15(15-12)(15-10)(15-8)}$ sq. m.

= $\sqrt{1575}$ sq. m.

= $39 \cdot 68$ sq. m. (approx.)

Again, area of the triangular region ABD = $\dfrac{1}{2} AB \times DF$

or, $39 \cdot 68 = \dfrac{1}{2} \times 12 \times DF$ or, $6 \cdot DF = 39 \cdot 68$; $\therefore$ $DF = 6 \cdot 61$

Now, in right angled triangle ΔBCE,

$$BE^2 = BC^2 - CE^2 = AD^2 - DF^2 = 8^2 - (6 \cdot 61)^2 = 20 \cdot 31$$

$\therefore \; BE = 4 \cdot 5$

Therefore, $AE = AB + BE = 12 + 4 \cdot 5 = 16 \cdot 5$

From right angled triangle ΔBCE, we get,

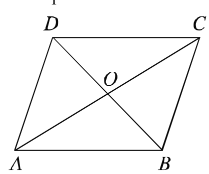
$AC^2 = AE^2 - CE^2 = (16 \cdot 5)^2 - (6 \cdot 61)^2 = 315 \cdot 94$

$\therefore \; AC = 17 \cdot 77$ (approx.)

$\therefore$ The required length of the diagonal is $17 \cdot 77$ m. (approx.)

Example 6. The length of a diagonal of a rhombus is 10m. and its area is 120 sq. m. Determine the length of the other diagonal and its perimeter.

Solution : Let, the length of a diagonal of rhombus $ABCD$ is $BD = d_1 = 10$ metre and another diagonal $AC = d_2$ metre.



$\therefore$ Area of the rhombus = $\frac{1}{2} d_1 d_2$ sq. m.

As per question, $\frac{1}{2} d_1 d_2 = 120$ or, $d_2 = \frac{120 \times 2}{10} = \frac{120 \times 2}{10} = 24$

We know, the diagonals of rhombus bisect each other at right angles. Let the diagonals interset at the point 0.

$\therefore \; OD = OB = \frac{10}{2}$ m. = 5 m. and $OA = OC = \frac{24}{2}$ m. = 12 m.

and in right angled triangle ΔAOD, we get

$AD^2 = OA^2 + OD^2 = 5^2 + (12)^2 = 169$; $\therefore \; AD = 13$

$\therefore$ The length of each sides of the rhombus is 13 m.

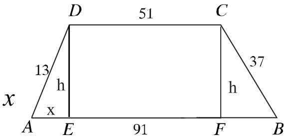
$\therefore$ The perimeter of the rhombus = 4×13 m. = 52 m..

The required length of the diagonal is 24 m. and perimeter is 52 m.

Example 7. The lengths of two parallel sides of a trapezium are 91 cm. and 51 cm. and the lengths of two other sides are 37 cm and 13 cm respectively. Determine the area of the trapezium.

Solution : Let, in trapezium $ABCD$; $AB = 91$ cm. $CD = 51$ cm. Let us draw the perpendiculars DF and CF on AB from D and C respectively.

$\therefore \; CDEF$ is a rectangle.



$\therefore \; EF = CD = 51$ cm.

Let, $AE = x$ and $DE = CF = h$

$\therefore \; BF = AB - AF = 91 - (AE + EF) = 91 - (x + 51) = 40 - x$

$\therefore$ From right angled triangle ΔADE, we get,

$AE^2 + DE^2 = AD^2$ or, $x^2 + h^2 = (13)^2$ or, $x^2 + h^2 = 169$.........(i)

Again, from right angled triangle ΔBCF, we get,

$BF^2 + CF^2 = BC^2$ or, $(40 - x)^2 + h^2 = (37)^2$

or, $1600-80x+x^2+h^2=1369$
or, $1600-80x+169=1396$ [with the help of (1)]
or, $1600+169-1396=80x$;or, $80x=400$; $\therefore$ $x=5$
Now putting the value of x in equation (1) we get
$5^2+h^2=163$ or, $h=169-25=144$; $\therefore$ $h=12$

Area of $ABCD$ = $\frac{1}{2}(AB+CD)\cdot h$

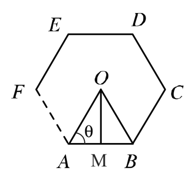
$= \frac{1}{2}(91+51)\times 12$ square cm.

= 852 square cm.

The required area is 852 square cm.

16·3 Area of regular polygon

The lengths of all sides of a regular polygon are equal. Again, the angles are also equal. Regular polygon with n sides produces n isocsceles triangles by adding centre to the vertices.



$\therefore$ Area of the regular polygon = $n\times$ area of one triangular region.

Let $ABCDEF$ be a regular polygon whose centre is O.

$\therefore$ It has n sides and the length of each side is a. We join O, A ; O, B.

$\therefore$ Let in ΔAOB height $OM=h$ and $\angle OAB=\theta$

$\therefore$ The angle produced at each of the vertices of regular polygon = 2θ .

$\therefore$ Angle produced by n number of vertices in the polygon = $2\theta\cdot n$

Angle produced in the polygon at the centre = 4 right angles.

The sum of angles of n number of triangles = $2\theta(n+4)$ right angles.

$\therefore$ Sum of 3 angles of ΔOAB = 2 right angles.

$\therefore$ The wise, summation of the angles of n numbers of triangles = $n\cdot 2$ right angles

$\therefore$ $2\theta(n+4)$ right angles = $n\cdot 2$ right angles

or, $2\theta\cdot n=(2n-4)$ right angles

or, $\theta=\frac{2n-4}{2n}$ right angles

or, $\theta=\left(1-\frac{2}{n}\right)$ right angles

or, $\theta=\left(1-\frac{2}{n}\right)\times 90^\circ = 90^\circ-\frac{180^\circ}{n}$

Now, $\tan\theta=\frac{h}{\frac{a}{2}}=\frac{2h}{a}$; $\therefore$ $h=\frac{a}{2}\tan\theta$

$\therefore$ Area of ΔOAB $= \frac{1}{2}ah$

$= \frac{1}{2}a \times \frac{a}{2}\tan\theta$

$= \frac{a^2}{4}\tan\left(90^\circ - \frac{180^\circ}{n}\right)$

$= \frac{a^2}{4}\cot\left(\frac{180^\circ}{n}\right)$

$\therefore$ Area of a regular polygon having n sides $= \frac{a^2}{4}\cot\left(\frac{180^\circ}{n}\right)$.

Example 8. If the length of each side of a regular pentagon is 4 cm, determine its area.

Solution : Let, length of each side of a regular pentagon is $a = 4$ cm.

and number of sides $n = 5$

We know, area of a regular polygon = $\frac{a^2}{4}\cot\frac{180^\circ}{n}$

$\therefore$ Area of the pentagon = $\frac{4^2}{4}\cot\frac{180^\circ}{5}$ Sq. cm.

$= 4\times\cot\ 36^o$ sq. cm.

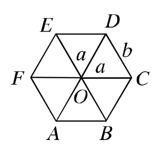
$= 4\times 1\cdot 376$ sq. cm. [with the help of calculator]

$= 5\cdot 506$ sq. cm. (approx.)

The required area = $5\cdot 506$ sq. cm. (approx.)

Example 9. The distance of the centre to the vertex of a regular hexagon is 4 m. Determine its area.

Solution : Let, $ABCDEF$ is a regular hexagon whose centre is O, O is joined to each of the vertex and thus 6 triangles of equal area are formed.



$\therefore\ \angle COD = \frac{360^\circ}{6} = 60^\circ$

Let the distance of centre O to its vertex is a m.

$\therefore$ a = 4.

$\therefore$ Area of ΔCOD $= \frac{1}{2}a\cdot a\sin 60^\circ = \frac{1}{2}a^2\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}a^2$

$= \frac{\sqrt{3}}{4}\times 4^2$ sq. m. $= 4\sqrt{3}$ sq. m.

$\therefore$ **Area of the regular hexagon**

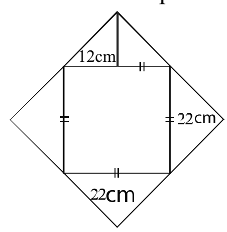
$= 6 \times 4\sqrt{3}$ sq, m.

$= 24\sqrt{3}$ sq. m.

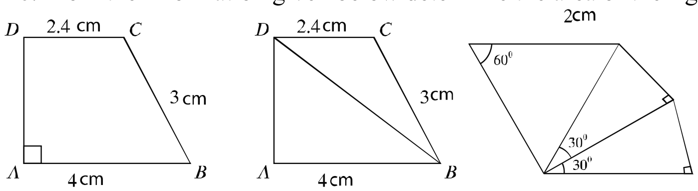
Exercise 16·2

1. The length of a rectangular region is twice its width. If its area is 512 sq. m., determine its perimeter.
2. The length of a plot is 80 m. and the breadth is 60 m. A rectangular pond was excavated in the plot. If the width of each side of the border around the pond is 4 metre, determine the area of the border of the pond.
3. The length of a garden is 40 metre and its breadth is 30 metre. There is a pond in side the garden with around border of equal width. If the area of the pond is $\frac{1}{2}$ of that of the garden, find the length and breadth of the pond.
4. Outside a square garden, there is a path 5 metre width around it. If the area of the path is 500 square metre, find the area of the garden.
5. The perimeter of a square region is equal to the perimeter of a rectangular region. The length of the rectangular region is thrice its breadth and the area is 768 sq. m. How many stones will be required to cover the square region with square stones of 40 cm each?
6. Area of a rectangular region is 160 sq. m. If the length is reduced by 6 m., it becomes a square region. Determine the length and the breadth of the rectangle.
7. The base of a parallelogram is $\frac{3}{4}$th of the height and area is 363 square inches. Determine the base and the height of the parallelogram.
8. The area of a parallelogram is equal to the area of a square region. If the base of the parallelogram is 125m. and the height is 5 m, find the length of the diagonal the square.
9. The length of two sides of a parallelogram are 30 cm and 26 cm, If the smaller diagonal is 28 cm, find the length of the other diagonal.
10. The perimeter of a rhombus is 180 cm. and the smaller diagonal is 54 cm. Find the other diagonal and the area.
11. Deference of the length of two parallel sides of a trapezium is 8 cm. and their perpendicular distance is 24 cm. Find the lengths of the two parallel sides of the trapezium.
12. The lengths of two parallel sides of a trapezium are 31 cm. and 11 cm. respectively and two other sides are 10 and 12 cm. respectively. Find the area of the trapezium.

13. The distance from the centre to the vertex of a regular octagon is 1.5 m. Find the area of the regular octagon.
14. The length of a rectangular flower garden is 150 m. and breadth is 100 m. For nursing the garden, there is a path with 3 m. width all along its length and breadth right at the middle of the garden.
    (a) Describe the above information with figure.
    (b) Determine the area of the path.
    (c) How many bricks of 25 cm. length and $12\cdot5$ cm. width will be required to make the path metalled ?
15. From the figure of the polygon determine its area.

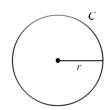


16. From the information given below determine the area of the figures :



6.4 Measurement regarding circle

(1) Circumference of a Circle



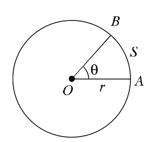
The length of a circle is called its circumference. Let r be the radius of a circle,
its circumference $c = 2\pi r$, where $\pi = 3\cdot14159265$.......... which is an irrational number value of $\pi = 3\cdot1416$ is used as the actual value.
Therefore, if the radius of a circle is known, we can find the approximate value of the circumference of the circle by using the value of π .

Example 1. The diameter of a circle is 26 cm. Find its circumference.

Solution : Let, the radius of the circle is r .
$\therefore$ diameter of the circle = $2r$ and circumference = $2\pi r$

As per question, $2r = 26$ or, $r = \frac{26}{2}$ $\therefore$ $r = 13$

$\therefore$ circumference of the circle = $2\pi r = 2\times3\cdot1416\times13$ cm.
$= 3\cdot1416\times26$ cm (approx.)



The required circumference of the circle is $81\cdot68$ cm. (approx.)

(2) Length of arc of a circle

Let O be the centre of a circle whose radius is r and are $AB = s$, which produces θ^o angle at the centre.

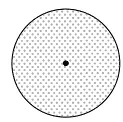
$\therefore$ Circumference of the circle = $2\pi r$

Total angle produced at the centre of the circle = 360^o and arc s produces angle $\theta°$ at the centre. We know, any interior angle at the centre of a circle produced by any arc is proportional to the arc.

$$\therefore \frac{\theta}{360} = \frac{s}{2\pi r} \text{ or, } s = \frac{\pi r\theta}{180}$$

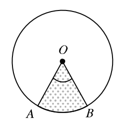
(3) Area of circular region and circular segment

The subset of the plane formed by the union of a circle and its interior is called a circular region and the circle is called the boundary of the such circular region.



Circular segment: The area formed by an arc and the radius related to the joining points of that arc is called circular segment.

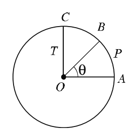
If A and B are two points on a circle with centre O, the subset of the plane formed by the union of the intersection of $\angle AOB$ and the interior of the circle with the line segment OA, OB and the arc AB, is called a circular segment.



In previous class, we have learnt that if the radius of a circle is r, the area is = πr^2

We know, any angle produced by an arc at the centre of a circle is proportional to the arc.

So, at this stage we can accept that the area of two circular segments of the same circle are proportional to the two arcs on which they stand.



Let us draw a radius r with centre O.

The circular segment AOB stands on the arc APB whose measurement is θ. Draw a perpendicular OC on OA.

$$\therefore \frac{\text{Area of circular segment } AOB}{\text{Area of circular segment } AOC} = \frac{\text{Measurement of } \angle AOB}{\text{Measurement of } \angle AOC}$$

$$\text{or, } \frac{\text{Area of circular segment } AOB}{\text{Area of circular segment } AOC} = \frac{\theta}{90} \; ; \; [\angle AOC = 90^o]$$

$$\text{or, Area of circular segment } AOB = \frac{\theta}{90} \times \text{ area of circular segment } AOC$$

$$= \frac{\theta}{90} \times \frac{1}{4} \times \text{ area of the circle}$$

$$= \frac{\theta}{90} \times \frac{1}{4} \times \pi r^2$$

$$= \frac{\theta}{360} \times \pi r^2$$

So, area of circular segment $= \dfrac{\theta}{360} \times \pi r^2$

Example 2. The radius of a circle is 8 cm. and a circular segment substends an angle 56^o at the centre. Find the length of the arc and area of the circular segment.

Solution : Let, radius of the circle, $r = 8$ cm, length of arc is s and the angle subtended by the circular segment is 56^o.

We know, $s = \dfrac{\pi r\theta}{180} = \dfrac{3 \cdot 1416 \times 8 \times 56}{180}$ cm.. $= 7 \cdot 82$ cm. (approx)

Area of circular segment $= \dfrac{\theta}{360} \times \pi r^2$

$$= \frac{56}{360} \times 3.1416 \times 8^2 \text{ sq.cm.}$$

$= 62 \cdot 55$ sq. cm. (approx)

Example 3. If the difference between the radius and circumference of a circle is 90 cm., find the radius of the circle.

Solution : Let the radius of the circle be r

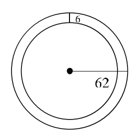
$\therefore$ Diameter of the circler is $2r$ and circumference $= 2\pi r$

As per question, $2\pi r - 2r = 90$

or, $2r(\pi - 1) = 90$ or, $r = \dfrac{90}{2(\pi - 1)} = \dfrac{45}{3 \cdot 1416 - 1} = 21 \cdot 01$ (approx.)

The required radius of the circle is $21 \cdot 01$ cm. (approx.).

Example 4. The diameter of a circular field is 124 m. There is a path with 6 m. width all around the field. Find the area of the path.



Solution : Let the radius of the circular field be r and radius of the field with the path be R.

$\therefore$ $r = \dfrac{124}{2}$ m. $= 62$ m. and $R = (62 + 6)$ m. $= 68$ m.

Area of the circular field $= \pi r^2$

and area of the circular field with the path $= \pi R^2$

$\therefore$ Area of the path = Area of field with path – Area of the field

$= (\pi R^2 - \pi r^2) = \pi(R^2 - r^2)$

$= 3 \cdot 1416\{(68)^2 - (62)^2\}$ sq. m.

$= 3 \cdot 1416(4624 - 3844)$

$= 3\cdot1416 \times 780$ sq. m.

$= 2450\cdot45$ sq. m. (approx)

The required area of the path is $2450\cdot44$ square m. (approx.)

Activity : Circumference of a circle is 440 m. Determine the length of the sides of the inscribed square in it.

Example 5. The radius of a circle is 12 cm. and the length of an arc is 14 cm. Determine the angle subtended by the circular segment at its centre.

Solution : Let, radius of the circle is r = 12 cm., the length of the arc is s = 14 cm. and the angle subtended at the centre is θ °.

We know, $s = \dfrac{\pi s\theta}{180}$

or, $\pi r\theta = 180 \times s$

or, $\theta = \dfrac{180 \times s}{\pi r} = \dfrac{180 \times 14}{3\cdot1416 \times 12} = 66\cdot85$ (approx)

$\therefore$ The required angle is 66·85° (approx)

Example 6. Diameter of a wheel is 4.5 m. for traversing a distance of 360 m.;how many times the wheel will revolve ?

Solution : Given that, the diameter of the wheel is 4.5 m.

$\therefore$ The radius of the wheel, $r = \dfrac{4\cdot5}{2}$ m. and circumference = $2\pi r$

Let, for traversing 360 m, the wheel will revolve n times

As per question, $n \times 2\pi r = 360$

or, $n = \dfrac{360}{2\pi r} = \dfrac{360 \times 2}{2 \times 3\cdot1416 \times 4\cdot5}$ = 25.46 (approx)

$\therefore$ The wheel will revolve 25 times (approx) for traversing 360 m..

Example 7. Two wheels revolve 32 and 48 times respectively to cover a distance of 211 m. 20 cm. Determine the difference of their radii.

Solution : 211 m. 20 cm. = 21120 cm.

Let, the radii of two wheels are R and r respectively ;where $R > r$.

$\therefore$ Circumferences of two wheels are $2\pi R$ and $2\pi r$ respectively and the difference of radii is $(R - r)$

As per question, $32 \times 2\pi R = 21120$

or, $R = \dfrac{21120}{32 \times 2\pi} = \dfrac{21120}{32 \times 2 \times 3\cdot1416} = 105\cdot04$ (approx)

and $48 \times 2\pi r = 21120$

or, $r = \dfrac{21120}{48 \times 2\pi} = \dfrac{21120}{48 \times 2 \times 3\cdot1416} = 70\cdot03$ (approx)

$\therefore\ R-r=(105\cdot04-70\cdot03)$ cm.. = $35\cdot01$ cm. = $\cdot35$ m. (approx)

$\therefore$ The difference of radii of the two wheels is $\cdot35$ m (approx)

Example 8. The radius of a circle is 14 cm. The area of a square is equal to the area of the circle. Determine the length of the square.

Solution : Let the radius of the circle, $r=14$ cm. and the length of the square is a

$\therefore$ Area of the square region = a^2 and the area of the circle = πr^2

According to the question, $a^2=\pi r^2$

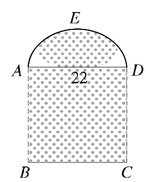
$\therefore$ Radius of the half circle $r=\frac{22}{2}$ m. = 11 m.

or, $a=\sqrt{\pi r}=\sqrt{3\cdot1416}\times14=24\cdot81$ (approx)

The required length is 24·81 cm. (approx.)

Example 9 : In the figure, $ABCD$ is a square whose length of each side is 22 m. and AED region is a half circle. Determine the area of the whole region.

Solution : Let, the length of each side of the square $ABCD$ be a.



$\therefore$ Area of square region = a^2

Again, AED is a half circle.

Let ;be its radius

$\therefore$ Area of the half circle, $AED=\frac{1}{2}\pi r^2$

$\therefore$ Area of the whole region = Area of the square $ABCD$ + area of the half circle AED.

$= a^2+\frac{1}{2}\pi r^2$

$=\{(22)^2+\frac{1}{2}\times3\cdot1416\times(11)^2\}$ sq. metre

[a = 22, r = $\frac{22}{2}$ = 11]

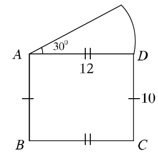
= 674.07 sq. m (app.)

The required area is 674.07 square metre (approx)

Example 10. In the figure, $ABCD$ is a rectangle whose the length is 12 m., the breadth is 10 m. and DAE is a circular region.

Determine the length of the arc DE and the area of the whole region.

Solution : Let the radius of the circular segment, $r=AD=12$ m. and the angle subtended at centre $\theta=30^\circ$



$\therefore$ length of the arc $DE=\frac{\pi r\theta}{180}$

$=\frac{3\cdot1416\times12\times30}{180}$ m. = 6.28 m. (approx.)

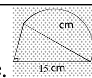
$\therefore$ Area of the circular segment DAE = $\frac{\theta}{360} \times \pi r^2 = \frac{30}{360} \times 3 \cdot 1416 \times (12)^2$ sq. m. = 37.7 sq.cm (approx)

The length of the rectangle $ABCD$ is 12 m. and the breadth is 10 m.

$\therefore$ Area of the rectangle = length × breadth = 12 × 10 sq. m = 120 sq. m.

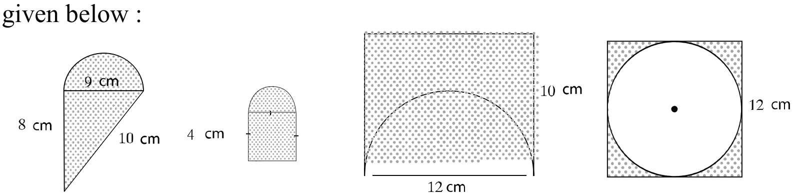
$\therefore$ Area of the whole region = (37.7 + 120) sq. m. = 157.7 square metre.

The required area is 157.7 square metre. (approx)

Activity : Determine the area of the dark marked region in the figure. 

Exercise 16.3

1. Angle subtended by a circular segment at the centre is 30°. If the diameter of the circle is 126 cm., determine the length of the arc.
2. A horse turned around a circular field with a speed of 66 m per minute in $1\frac{1}{2}$ minute. Determine the diameter of the field.
3. Area of a circular segment is 77 sq. m. and the radius is 21 m. Determine the angle subtended at the centre by the circular segment.
4. The radius of a circle is 14 cm. and an arc subtends an angle 76° at its centre. Determine the area of the circular segment.
5. There is a road around a circular field. The outer circumference of the road is greater than the inner circumference by 44 metres. Find the width of the road.
6. The diameter of a circular park is 26 m. There is a road of 2 m. width around the park outside. Determine the area of the road.
7. The diameter of the front wheel of a car is 28 cm. and the back wheel is 35 cm. To cover a distance of 88 m, how many times more the front wheel will revolve than the back one ?
8. The circumference of a circle is 220 m. Determine the length of the side of the inscribed square in the circle.
9. The circumference of a circle is equal to the perimeter of an equilateral triangle. Determine the ratio of their areas ?
10. Determine the area of the dark marked region with the help of the information given below :

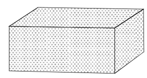


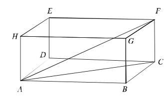
6·5 Rectangular solid

The region surrounded by three pairs of parallel rectangular planes is known as rectangular solid.

Let, $ABCDEFGH$ is a rectangular solid, whose length $AB = a$, and breadth $BC = b$ and height $AH = c$

(1) Determining the diagonal: AF is the diagonal of the rectangular solid $ABCDEFGH$





In ΔABC, $BC \perp AB$ and AC is hypotenuse

$\therefore\ AC^2 = AB^2 + BC^2 = a^2 + b^2$

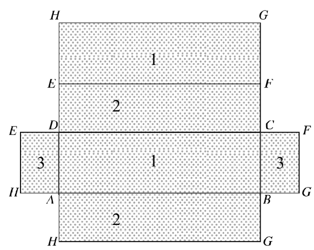
Again, in ΔACF, $FC \perp AC$ and AF is hypotenuse

$\therefore\ AF^2 = AC^2 + CF^2 = a^2 + b^2 + c^2$

$\therefore\ AF = \sqrt{a^2 + b^2 + c^2}$

$\therefore$ the diagonal of the rectangular solid $= \sqrt{a^2 + b^2 + c^2}$

(2) Determination of area of the whole surface :



There are 6 surfaces of the rectangular solid where the opposite surfaces are equal figure

Area of the whole surface of the rectangular solid

$= 2$ (area of the surface of $ABCD$ + area of the surface of $ABGH$ + area of the surface of $BCFG$)

$= 2(AB \times AD + AB \times AH + BC \times BG)$

$= 2(ab + ac + bc)$

$= 2(ab + bc + ca)$

(3) Volume of the rectangular solid = length $\times$ width $\times$ height

$= abc$

Example 1. The length, width and height of a rectangular solid are 25 cm., 20 cm. and 15 cm. respectively. Determine its area of the whole surface, volume and the length of the diagonal.

Solution : Let, the length of the rectangular solid is $a = 25$ cm., width $b = 20$ cm. and height $c = 15$ cm.

$\therefore$ Area of the whole surface of the rectangular solid $= 2(ab + bc + ca)$

$= 2(25 \times 20 + 20 \times 15 + 15 \times 25)$ sq. cm.

$= 2350$ square cm.

Volume $= abc$

$= 25 \times 20 \times 15$ cube cm.

$= 7500$ cube cm.

And the length of its diagonal $= \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{(25)^2 + (20)^2 + (15)^2}$ cm.

$= \sqrt{625 + 400 + 225}$ cm.

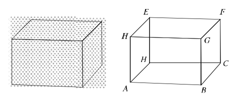
$= \sqrt{1250}$ cm.

$= 35 \cdot 353$ cm. (approx.)

The required area of the whole surface is 2350 cm^2., volume 7500 cm^3. and the length of the diagonal is $35 \cdot 353$ cm. (approx.).

Activity : Determine the volume, area of the whole surface and the length of the diagonal of your mathematics book calculating its length, width and height.

6·6 Cube



If the length, width and height of a rectangular solid are equal, it is called a cube.

Let, $ABCDEFGH$ is a cube.

Its height = width = height = a

(1) The length of diagonal of the cube = $\sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = \sqrt{3}a$

(2) The area of the whole surface of the cube $= 2(a \cdot a + a \cdot a + a \cdot a)$

$= 2(a^2 + a^2 + a^2) = 6a^2$

(3) The volume of the cube = $a \cdot a \cdot a = a^3$

Example 2. The area of the whole surface of a cube is 96 m^2. Determine the length of its diagonal.

Solution : Let, the sides of the cube is a

$\therefore$ The area of its whole surface $= 6a^2$ and the length of diagonal = $\sqrt{3}a$

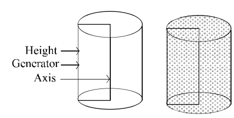
As per question, $6a^2 = 96$ or, $a^2 = 16$; $\therefore a = 4$

$\therefore$ The length of diagonal of the cube = $\sqrt{3}a = \sqrt{3} \times 4 = 6 \cdot 928$ m. (approx.)

The required length of the diagonal is $6 \cdot 928$ m. (approx.)

Activity : The sides of 3 metal cube are 3 cm., 4 cm. and 5 cm. respectively. A new cube is formed by melting the 3 cubes. Determine the area of the whole surface and the length of the diagonal of new cube.

6·7 Cylinder :



The solid formed by a complete revolution of any rectangle about one of its sides as axis is called a cylinder or a right circular cylinder. The two ends of a right circular cylinder are circles. The curved face is called curved surface and the total plane is called whole surface. The side of the rectangle which is parallel to the axis and revolves about the axis is called the generator line of the cylinder.

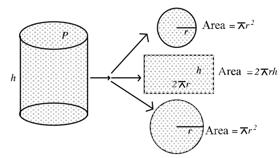
Let, figure (a) is a right circular cylinder, whose radius is r and height h

(1) Area of the base = πr^2

(2) Area of the curved surface

= perimeter of the base × height

= $2\pi rh$



(3) Area of the whole surface

$= (\pi r^2 + 2\pi rh + \pi r^2) = 2\pi r(r+h)$

(4) The volume = area of the base × height

$= \pi r^2 h$

Example 3. If the height of a right circular cylinder is 10 cm. and radius of the base is 7 cm, determine its volume and the area of the whole surface.

Solution : Let, the height of the right circular cylinder is $h = 10$ cm. and radius of the base is r.

$\therefore$ Its volume $= \pi r^2 h = 3\cdot1416 \times 7^2 \times 10$

$= 1539\cdot38$ cube cm. (approx.)

And the area of the whole surface = $2\pi r(r+h)$

$= 2 \times 3\cdot1416 \times 7(7+10)$ sq. cm. (approx.)

$= 747\cdot7$ sq. cm. (approx.)

Activity : Make a right circular cylinder using a rectangular paper. Determine the area of its whole surface and the volume.

Example 4. The outer measurements of a box with its top are 10 cm., 9 cm. and 7cm. respectively and the area of the whole inner surface is 262 cm^2. Find the thickness of its wall if it is uniform on all sides.

Solution : Let, the thickness of the box is x cm.

The outer measurements of the box with top are 10 cm., 9 cm. and 7 cm. respectively.

$\therefore$ The inside measurement of the box are respectively $a = (10-2x)$ cm., $b = (9-2x)$ cm. and $c = (7-2x)$ cm.

$\therefore$ The area of the whole surface of the inner side of the box = $2(ab+bc+ca)$

As per question, $2(ab+bc+ca) = 262$

or, $(10-2x)(9-2x)+(9-2x)(7-2x)+(7-2x)(10-2x) = 131$

or, $90-38x+4x^2+63-32x+4x^2+70-34x+4x^2-131 = 0$

or, $12x^2-104x+92 = 0$

or, $3x^2-26x+23 = 0$

or, $3x^2-3x-23x+23 = 0$

or, $3x(x-1)-23(x-1) = 0$

or, $(x-1)(3x-23) = 0$

or, $x-1 = 0$ or, $3x-23 = 0$

or, $x = 1$ or, $x = \frac{23}{3} = 7\cdot67$ (approx.)

But the thickness of a box cannot be greater than or equal to the length or width or height

$\therefore\ x = 1$

The required thickness of the box is 1 cm.

Example 5. If the length of diagonal of the surface of a cube is $8\sqrt{2}$ cm., determine the length of its diagonal and volume.

Solution : Let, the side of the cube is a.

$\therefore$ The length of diagonal of the surface = $\sqrt{2}a$

Length of diagonal = $\sqrt{3}a$

And the volume $= a^3$

As per question, $\sqrt{2}a = 8\sqrt{2}$; $\therefore\ a = 8$

$\therefore$ The length of the cube's diagonal = $\sqrt{3}\times 8$ cm. = $8\sqrt{3}$ cm.

And the volume = 8^3 cm^3. = 512 cm^3.

The required length of the diagonal is $8\sqrt{3}$ cm. (approx) and the volume is 512 cm^3.

Example 6. The length of a rectangle is 12 cm. and width 5 cm. If it is revolved around the greater side, a solid is formed. Determine the area of its whole surface and the volume.

Solution : Given that, the length of a rectangle is 12 cm. and width 5 cm. If it is revolved around the greater side, a circle based right cylindrical solid is formed with height $h = 12$ cm. and radius of the base $r = 5$ cm.

$\therefore$ The whole surface of the produced solid = $2\pi r(r+h)$

$= 2\times 3\cdot 1416\times 5(5+12)$ sq. sm.

$= 534\cdot 071$ sq. cm. (approx.)

And the volume = $\pi r^2 h$

$= 3\cdot 1416\times 5^2\times 12$ cm^3.

$= 942\cdot 48$ cube cm^3. (approx.)

The required area of whole surface is $534\cdot 071$ sq. cm. (approx.) and the volume is $942\cdot 48$ cm^3. (approx.)

Exercise 16·4

1. The length and witdth of two adjacent sides of a parallelogram are 7 cm., and 5 cm. respectively. What is the half of its perimeter in cm.. ?

   (a) 12 (b) 20 (c) 24 (d) 28

2. The length of the side of an equilateral triangle is 6 cm. What is its area (cm^2) ?

   (a) $3\sqrt{3}$ (b) $4\sqrt{3}$ (c) $6\sqrt{3}$ (d) $9\sqrt{3}$

3. If the height of a trapezium is 8 cm. and the lengths of the parallel sides are 9 cm. and 7 cm. respectively, what is its area (cm^2) ?

   (a) 24 (b) 64 (c) 96 (d) 504

4. Follow the information given below :

(*i*) A square stone with the side of 4 cm. has 16 cm. perimeter.

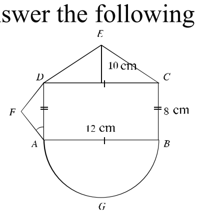
(*ii*) The area of circular sheet with the radius 3 cm.is 3π cm^2.

(*iii*) The volume of a cylinder with height of 5cm. and the radius of 2 cm. is 20π cm^3.

According to the information above, which one of the following is correct ?

(a) *i* and *ii* (b) *i* and *iii* (c) *ii* and *iii* (d) *i* , *ii* and *iii*

Answer the following questions (5 – 7) as per information from the picture below:



5. What is the length of the diagonal of the rectangle $ABCD$ in cm. ?
   (a) 13 (b) 14 (c) $14\cdot4$ (app.) (d) 15
6. What is the area of the circular segment ADF in sq. cm.?
   (a) 16 (b) 32 (c) 64 (d) 128
7. What is the circumference of the half circle AGB in cm. ?
   (a) 18 (b) $18\cdot85$ (app.) (c) $37\cdot7$ (app.) (d) 96
8. The length, width and height of a rectangular solid are 16 m. 12 m. and $4\cdot5$ m. respectively. Determine the area of its whole surface, length of the diagonal and the volume.
9. The ratios of the length, width and height of a rectangular solid are $21:16:12$ and the length of diagonal is 87 cm. Determine the area of the whole surface of the solid.
10. A rectangular solid is standing on a base of area 48 m^2 Its height is 3m and diagonal is 13 m. Determine the length and width of the rectangular solid.
11. The outer measurements of a rectangular wooden box are 8 cm., 6 cm. and 4 cm., respectively and the area of the whole inner surface is 88 cm^2. Find the thickness of the wood of the box.
12. The length of a wall is 25 m, height is 6 m. and breadth is 30 cm. The length, breadth and height of a brick is 10 cm. 5 cm. and 3 cm. respectively. Determine the number of bricks to build the wall with the bricks.
13. The area of the surface of a cube is 2400 sq. cm. Determine the diagonal of the cube.
14. The radius and the height of a right circular cylinder are 12 cm. and 5 cm. respectively. Find the area of the curved surface and the volume of the cylinder.
15. The area of a curved surface of a right circular cylinder is 100 sq. cm. and its volume is 150 cubic cm. Find the height and the radius of the cylinder.

16. The area of the curved surface of a right circular cylinder is 4400 sq. cn. If its height is 30 cm., find the area of its whole surface.
17. The inner and outer diameter of a iron pipe is 12 cm. and 14 cm. respectively. If the height of the pipe is 5 m., find the weight of the iron pipe where weight of 7.2 gm. iron $= 1$ cm ^{3}.
18. The length and the breadth of a rectangular region are 12 m. and 5m. respectively. There is a circular region just around the rectangle. The places which is are not occupied by the rectangle, are planted with grass.
    (a) Describe the information above with a figure.
    (b) Find the diameter of the circular region.
    (c) If the cost of planting grass per sq. m. is Tk. 50, find the total cost.
19. ΔABC and ΔBCD are on the same base BC and on the same parallel lines BC and AD.
    (a) Draw a figure as per the description above.
    (b) Proof that Δ region ABC = Δ region BCD.
    (c) Draw a parallelogram whose area is equal to the area of ΔABC and whose one of the angles is equal to a given angle (construction and description of construction is must).
20. $ABCD$ is a parallelogram and $BCEF$ is a rectangle and BC is the base of both of them.
    (a) Draw a figure of the rectangle and the parallelogram assuming the same height.
    (b) Show that the perimeter of $ABCD$ is greater than the perimeter of $BCEF$.
    (c) Ratio of length and width of the rectangle is 5 : 3 and its perimeter is 48 m. Determine the area of the parallelogram

Chapter Seventeen

Statistics

Owing to the contribution of information and data, the world has become a global village for the rapid advancement of science and information. Globalization has been made possible due to rapid transformation and expansion of information and data. So, to keep the continuity of development and for participating and contribute in globalizations, it is essential for the students at this stage to have clear knowledge about information and data. In the context, to meet the demands of students in acquiring knowledge, information and data have been discussed from class V and class-wise contents have been arranged step by step. In continuation of this, the students of this class will know and learn cumulative frequency, frequency polygon, ogive curve in measuring of central tendency mean, median, mode etc. in short-cut method.

At the end of this chapter, the students will be able to -

- Explain cumulative frequency, frequency polygon and ogive curve;
- Explain data by the frequency polygon, and ogive curve ;
- Explain the method of measuring of central tendency ;
- Explain the necessity of short-cut method in the measurement of central tendency ;
- Find the mean, median and mode by the short-cut method ;
- Explain the diagram of frequency polygon and ogive curve.

Presentation of Data : We know that numerical information which are not qualitative are the data of statistics. The data under investigation are the raw materials of statistics. They are in unorganized form and it is not possible to take necessary decision directly from the unorganized data. It is necessary to organize and tabulate the data. And the tabulation of data is the presentation of the data. In previous class we have learnt how to organize the data in tabulation. We determine know that it is required to the range of data for tabulation. Then determining the class interval and the number of classes by using tally marks, the frequency distribution table is made. Here, the methods of making frequency distribution table are to be re-discussed through example for convenient understanding.

Example 1. In a winter season, the temperature (in celsius) of the month of January in the district of Srimangal is placed below. Find the frequency distribution table of the temperature.

14°, 14°, 14°, 13°, 12°, 13°, 10°, 10°, 11°, 12°, 11°, 10°, 9°, 8°, 9°,
11°, 10°, 10°, 8°, 9°, 7°, 6°, 6°, 6°, 6°, 7°, 8°, 9°, 9°, 8°, 7°.

Solution : The minimum and maximum numerical values of the data of temperature are 6 and 14 respectively.
Hence the range = 14 – 6 + 1 = 9.

If the class interval is considered to be 3, the numbers of class will be $\frac{9}{3}$ or, 3.

Considering 3 to be the class interval, if the data are arranged in 3 classes, the frequency table will be :

Temperature (in celcius)	**Tally**	**Frequency**
6° – 8°	~~||||~~ ~~||||~~ |	11
9° – 11°	~~||||~~ ~~||||~~ |||	13
12° – 14°	~~||||~~ ||	7
		Total = 31

> **Activity :** Form two groups of all the students studying in your class. Find the frequency distribution table of the weights (in Kgs) of all the members of the groups.

Cumulative Frequency :

In example1, considering 3 the class interval and determining the number of classes, the frequency distribution table has been made. The numbers of classes of the mentioned data are 3. The limit of the first class is 6° – 8°. The lowest range of the class is 6° and the highest range is 8°C. The frequency of this class is 11.

The frequency of the second class is 13. Now if the frequency 11 of first class is added to the frequency 13 of the second class, we get 24. This 24 will be the cumulative frequency of the second class and the cumulative frequency of first class as begins with the class will be 11. Again, if the cumulative frequency 24 of the second class is added to the frequency of the third class, we get 24 + 7 = 31 which is the cumulative frequency of the third class. Thus cumulative frequency distribution table is made. In the context of the above discussion, the cumulative frequency distribution of temperature in example 1 is as follow :

Temperature (in celsius)	**Frequency**	**Cumulative Frequency**
6° – 8°	11	11
9° – 11°	13	(11 + 13) = 24
12° – 14°	7	(24 + 7) = 31

Example 2. The marks obtained in English by 40 students in an annual examination are given below. Make a cumulative frequency table of the marks obtained.

70, 40, 35, 60, 55, 58, 45, 60, 65, 80, 70, 46, 50, 60, 65, 70, 58, 69, 48, 70, 36, 85, 60, 50, 46, 65, 55, 61, 72, 85, 90, 68, 65, 50, 40, 56, 60, 65, 46, 76.

Solution : Range of the data = (highest numerical value − lowest numerical value) + 1

= (90 − 35) + 1

= 55 + 1

= 56

Let the class interval be 5, the number of classes = $\frac{56}{5}$

= 11·2 or 12

Hence the cumulative frequency distribution table at a class interval of 5 will be as follow :

Obtained marks	Frequency	Cumulative frequency	Obtained marks	Frequency	Cumulative frequency
35 – 39	2	2	70 – 74	4	4 + 31 = 35
40 – 44	2	2 + 2 = 4	75 – 79	1	1 + 35 = 36
45 – 49	5	5 + 4 = 9	80 – 84	1	1 + 36 = 37
50 – 54	3	3 + 9 = 12	85 – 89	2	2 + 37 = 39
55 – 59	5	5 + 12 = 17	90 – 94	1	1 + 39 = 40
60 – 64	8	8 + 17 = 25	95 – 99	0	0 + 40 = 40
65 – 69	6	6 + 25 = 31			

Variable : We know that the numerical information is the data of statistics. The numbers used in data are variable. Such as, the numbers indicating temperatures are variable. Similarly, in example 2, the secured marks used in the data are the variables.

Discrete and Indiscrete Variables : The variables used in statistics are of two types. Such as, discrete and indiscrete variables. The variables whose values are only integers, are discrete variables. The marks obtained in example 2 are discrete variables. Similarly, only integers are used in population indicated data. That is why, the variables of data used for population are discrete variables. And the variables whose numerical values can be any real number are indiscrete variables. Such as, in example 1, the temperature indicated data which can be any real number. Besides, any real number can be used for the data related to age, height, weight etc. That is why, the variables used for those are indiscrete variables. The number between two indiscrete variables can be the value of those variables. Some times it becomes necessary to make class interval indiscrete. To make the class interval indiscrete, the actual higher limit of a class and the lower limit of the next class are determined by fixing mid-point of a higher limit of any class and the lower limit of the next class. Such as, in example 1 the actual higher-lower limits of the first class are 8·5° and 5·5° respectively and that of the second class are 11·5° and 8·5° etc.

Activity : Form a group of maximum 40 students of your class. Form frequency distribution table and cumulative frequency table of the group with the weights/heights of the members.

Diagram of Data : We have seen that the collected data under investigation are the raw materials of the statistics. If the frequency distribution and cumulative frequency distribution table are made with them, it becomes clear to comprehend and to draw a conclusion. If that tabulated data are presented through diagram, they become easier to understand as well as attractive. That is why, presentation of statistical data in tabulation and diagram is widely and frequently used method. In class VIII, different types of diagram in the form of line graph and histogram have been discussed elaborately and the students have been taught how to draw them. Here, how frequency polygon, pie-chart, ogive curve drawn from frequency distribution and cumulative frequency table will be discussed.

Frequency Polygon : In class VIII, we have learnt how to draw the histogram of discrete data. Here how to draw frequency polygon from histogram of indiscrete data will be put for discussion through example.

Example 3. The frequency distribution table of the weights (in kg) of 60 students of class X of a school are is follows :

Weight (in kg)	46 – 50	51 – 55	56 – 60	61 – 65	66 – 70
Frequency (No of Students)	5	10	20	15	10

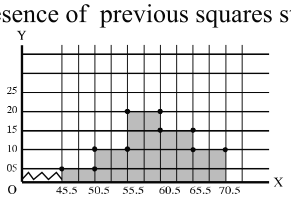
(a) Draw the histogram of frequency distribution.

(b) Draw frequency polygon of the histogram.

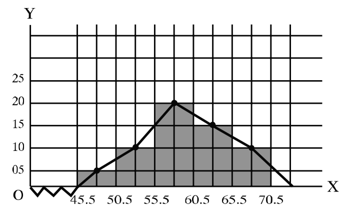
Solution : The class interval of the data in the table is discrete. If the class interval are made indiscrete, the table will be :

Class interval of the weight (in kg)	Discrete class interval	Mid point of class	Frequency
46 – 50	45·5 – 50·5	48	5
51 – 55	50·5 – 55·5	53	10
56 – 60	55·5 – 60·5	58	20
61 – 65	60·5 – 65·5	63	15
66 – 70	65·5 – 70·5	68	10

(a) Histogram has been drawn taking each square of graph paper as unit of class interval along with x-axis and frequency along with y-axis. The class interval along with x-axis has started from. The broken segments have been used to show the presence of previous squares starting from from origin to 45·5.



(b) The mid-points of the opposite sides parallel to the base of rectangle of the histogram have been fixed for drawing frequency polygon from histogram. The mid-points have been joined by line segments to draw the frequency polygon (shown in the adjacent figure). The mid-points of the first and the last rectangles have been joined with x-axis representing the class interval by the end points of line segments to show the frequency polygon attractive.

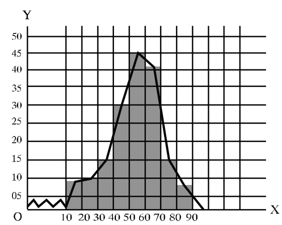


Frequency Polygon : The diagram drawn by joining frequency indicated points opposite to the class interval of indiscrete data by line segments successively is frequency polygon.

Example 4. Draw polygon of the following frequency distribution table :

Class interval	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90
Mid-point	15	25	35	45	55	65	75	85
Frequency	8	10	15	30	45	41	15	7

Solution : Histogram of frequency distribution is drawn taking two squares of graph paper as 5 units of class interval along with x-axis and 2 squares of graph paper as 5 units of frequency along with y-axis. The mid-points of the sides opposite to the base of rectangle of histogram are identified which are the mid-points of the class. Now the fixed mid-points are joined. The end-points of the first and the last classes are joined to x-axis representing the class interval to draw frequency polygon.



Activity : Draw frequency polygon from the marks obtained in Bangla by the students of your class in first terminal examination.

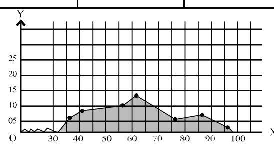
Example 5. The frequency distribution table of the marks obtained by 50 students of class X in science are given. Draw the frequency polygon of the data (without using histogram) :

Class interval of marks obtained	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Frequency	6	8	10	12	5	7	2

Solution : Here the given data are discrete. In this case, it is convenient to draw frequency polygon directly by finding the mid-point of class interval.

Class interval	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Mid-point	$\frac{40+31}{2} = 35\cdot5$	45·5	55·5	65·5	75·5	85·4	95·5
Frequency	6	8	10	12	5	7	2

The polygon is drawn by taking 2 squares of graph paper as 10 units of mid-points of class interval along with x-axis and taking two squares of graph paper as one units of frequency along with y-axis.



Activity : Draw frequency polygon from the frequency distribution table of heights of 100 students of a college.

Heights (in cm.)	141–150	151–160	161–170	171–180	181–190
Frequency	5	16	56	11	4

Cumulative Frequency Diagram or Ogive curve : Cumulative frequency diagram or Ogive curve is drawn by taking the upper limit of class interval along with x-axis and cumulative frequency along with y-axis after classification of a data

Example 6. The frequency distribution table of the marks obtained by 50 students out of 60 students is as follow :

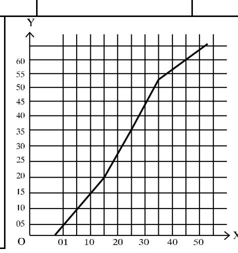
Class interval of marks obtained	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	8	12	15	18	7

Draw the Ogive curve of this frequency distribution.

Solution : The cumulative frequency table of frequency distribution of the given data is :

Class interval of marks obtained	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	8	12	15	18	7
Cumulative frequency	8	$8+12=20$	$15+20=35$	$18+35=53$	$7+53=60$

Ogive curve of cumulative frequency of data is drawn taking two squares of graph paper as unit of upper limit of class interval along with x-axis and one square of graph paper as 5 units of cumulative frequency along with y-axis.



Activity : Make cumulative frequency table of the marks obtained 50 and above in Mathematics by the students of your class in an examination and draw an Ogive curve.

Central Tendency : Central tendency and its measurement have been discussed in class VI and VII. We have seen if the data under investigation are arranged in order of values, the data cluster round near any central value. Again if the disorganized data are placed in frequency distribution table, the frequency is found to be abundant in a middle class i.e. frequency is maximum in middle class. In fact, the tendency of data to be clustered around the central value is number and it represents the data. The central tendency is measured by this number. Generally, the measurement of central tendency is of three types (1) Arithmetic means (2) Median (3) Mode :

Arithmetic Mean : We know if the sum of data is divided by the numbers of the data, we get the arithmetic mean. But this method is complex, time consuming and there is every possibility of committing mistake for large numbers of data. In such cases, the data are tabulated through classification and the arithmetic mean is determined by short-cut method.

Example 7. The frequency distribution table of the marks obtained by the students of a class is as follows. Find the arithmetic mean of the marks.

Class interval	25–34	35–44	45–54	55–64	65–74	75–84	85–94
Frequency	5	10	15	20	30	16	4

Solution : The class interval is given and that is why it is not possible to know the individual marks of the students. In such case, it becomes necessary to know the mid-value of the class.

Mid-value of the class = $\frac{\text{Class upper value} + \text{class lower value}}{2}$

If the class mid-value is $x_1 (i = 1, \text{.......}, k)$, the mid-value related table will be as follows:

Class interval	**Class mid-value** x_i	**Frequency** f_i	$f_i x_i$
25 – 34	29·5	5	147·5
35 – 44	39·5	10	395·0
45 – 54	49·5	15	742·5
55 – 64	59·5	20	1190·0
65 – 74	69·5	30	2085·0
75 – 84	79·5	16	1272·0
85 – 94	89·5	4	358·0
	Total	100	6190·0

The required mean = $\frac{1}{n}\sum_{i=1}^{k} f_i x_i$

$= \frac{1}{100} \times 6190$

$= 61 \cdot 9.$

Arithmetic mean of classified data (short-cut method)

The short-cut method is easy for determining arithmetic mean of classified data.

The steps to determine mean by short-cut method are :

1. To find the mid-value of classes.
2. To take convenient approximated mean (a) from the mid-values.
3. To determine steps deviation, the difference between class mid-values and approximate mean are divided by the class interval i.e.

   u = $\dfrac{\text{mid value} - \text{approximate mean}}{\text{class interval}}$
4. To multiply the steps deviation by the corresponding class frequency.
5. To determine the mean of the deviation and to add this mean with approximate mean to find the required mean.

Short-cut method : The formula used for determining the mean of the data by this method is $\bar{x} = a + \frac{1}{n}\sum f_i u_i \times h$ where $\bar{x}$ is required mean, a is approximate mean, The f_i is class frequency of ith class, $u_i f_i$ is the product of step deviation with class intervals of ith class and h is class interval.

Example 8. The production cost (in hundred taka) of a commodity at different stages is shown in the following table. Find the mean of the expenditure by short-cut method.

Production cost (in hundred taka)	2-6	6-10	10-14	14-18	18-22	22-26	26-30	30-34
Frequency	1	9	21	47	52	36	19	3

Solution : To determine mean in the light of followed steps in short-cut method, the table will be :

Class interval	**Mid-value** x_i	**Frequency** f_i	**Step deviation** $u_i = \frac{x_i - a}{h}$	**Frequency and class interval** $f_i u_i$
2 – 6	4	1	– 4	– 4
6 – 10	8	9	– 2	– 27
10 – 14	12	21	– 3	– 42
14 – 18	16	47	– 1	– 47
18 – 22	20 a	52	0	0
22 – 26	24	36	1	36
26 – 30	28	19	2	38
30 – 34	32	3	3	9
Total		188		– 37

Mean $\bar{x} = a + \frac{\sum f_i u_i}{n} \times h$

$$= 20 + \frac{-37}{188} \times 4$$

$$= 20 - \cdot 79$$

$$= 19 \cdot 22$$

$\therefore$ Mean production cost is Tk. 19·22 hundred.

Weighted mean : In many cases the numerical values $x_1, x_2, \ldots\ldots, x_n$ of statistical data under investigation may be influenced by different reasons /importance /weight. In such case, the values of the data $x_1, x_2, \ldots\ldots x_n$ along with their reasons/importance /weight $w_1, w_2, \ldots\ldots w_n$ are considered to find the arithmetic mean.

If the values of n numbers of data are $x_1, x_2, \ldots\ldots x_n$ and their weights are $w_1, w_2, \ldots\ldots w_n$, the weighted mean will be

$$\overline{x_w} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}$$

Example 9. The rate of passing in degree honours class and the number of students of some department of a University are presented in the table below. Find the mean rate of passing in degree honours class of those departments of the university.

Name of the department	Math	Statistics	English	Bangla	Zoology	Pol. Science
Rate of passing (in percentage)	70	80	50	90	60	85
Number of Students	80	120	100	225	135	300

Solution : Here, the rate of passing and the number of students are given. The weight of rate of passing is the number of students. If the variables of rate of passing are x and numerical variable of students is w, the table for determining the arithmetic mean of given weight will be as follows :

Department	x_i	w_i	$x_i \; w_i$
Math	70	80	5600
Statistics	80	120	9600
English	50	100	5000
Bangali	90	225	20250
Zoology	60	135	8100
Pol. Science	85	300	25500
Total		960	74050

$$\overline{x_w} = \frac{\sum_{i=1}^{6} x_i w_i}{\sum_{i=2}^{6} w_i} = \frac{74050}{960} = 77 \cdot 14$$

Mean rate of passing is 77·14

Activity : Collect the rate of passing students and their numbers in SSC. examination of some schools in your zilla and find mean rate of passing.

Median

We have already learnt in class VII the value of the data which divide the data when arranged in ascending order into two equal parts are median of the data. We have also learnt if the numbers of data are n and n is an odd number, the median will be the value of $\frac{n+1}{2}$ th term. But if n is an even number, the median will be numerical mean of the value of $\frac{n}{2}$ and $\left(\frac{n}{2}+1\right)$ th terms. Here we present through example how mean is determined with or without the help formulae.

Example 10. The frequency distribution table of 51 students is placed below. Find the median.

Height (in cm.)	150	155	160	165	170	175
Frequency	4	6	12	16	8	5

Solution : Frequency distribution table for finding mean is an follows :

Height (in cm.)	150	155	160	165	170	175
Frequency	4	6	12	16	8	5
Cumulative Frequency	4	10	22	38	46	51

Here, $n = 51$ which is an odd number.

$\therefore$ Median = the value of $\frac{51+1}{2}$ th term

= the value of 26 th term = 165

Required median is 165 c.m.

Note : The value of the terms from 23th to 38th is 165.

Example 11. The frequency distribution table of marks obtained in mathematics of 60 students is as follows. Find the median :

Marks obtained	40	45	50	55	60	70	80	85	90	95	100
Frequency	2	4	4	3	7	10	16	6	4	3	1

Solution : Cumulative frequency distribution table for determining median is :

Marks obtained	40	45	50	55	60	70	80	85	90	95	100
Frequency	2	4	4	3	7	10	16	6	4	3	1
Cumulative frequency	2	6	10	13	20	30	46	52	56	59	60

Here, $n = 60$ which is an even number.

$$\therefore \text{Median} = \frac{\text{The sum of values of } \frac{60}{2} \text{th and } \frac{60}{2}+1 \text{th terms}}{2}$$

$$= \frac{\text{The sum of values of 30th and 31th terms}}{2}$$

$$= \frac{70+80}{2} = \frac{150}{2} = 75$$

$\therefore$ Required Median is 75.

Activity : 1. Make frequency distribution table of the heights (in cm.) of 49 students of your class and find the mean without using any formula. 2. From the above problem, deduct the heights of 9 students and then find the median of heights (in cm.) of 40 students.

Determining Median of Classified Data

If the number of classified data is n, the value of $\frac{n}{2}$th term of classified data is median. And the formula used to determine the median or the value of $\frac{n}{2}$th term is :

Median = $L+\left(\frac{n}{2}-F_c\right)\times\frac{h}{f_m}$, where L is the lower limit of the median class, n is the frequency, F_c is the cumulative frequency of previous class to median class, f_m is the frequency of median class and h is the class interval.

Example 12. Determine median from the following frequency distribution table :

Time (in sec.)	30–35	36–41	42–47	48–53	54–59	60–65
Frequency	3	10	18	25	8	6

Solution : Frequency distribution table for determining median :

Time (in sec.) (class interval)	**Frequency**	**Cumulative Frequency**
30 – 35	3	3
36 – 41	10	13
42 – 47	18	31
48 – 53	25	56
54 – 59	8	64
60 – 65	6	70
	n = 70	

Here, $n = 70$ and $\frac{n}{2} = \frac{70}{2}$ or 35.

Therefore, median is the value of 35th term. 35th term lies in the class (48 – 53). Hence the median class is (48 – 53).

Therefore, $L = 48,\ F_c = 31,\ F_m = 25$ and $h = 6$.

Median $= L+\left(\frac{n}{2}-F_c\right)\times\frac{h}{f_m}$

$= 48 + (35 - 31) \times \frac{6}{25} = 48 + 4 \times \frac{6}{25}$

$= 48 + 0 \cdot 96$

$= 48 \cdot 96$

Required median is 48·96

Activity : Make two groups with all the students of your class. (a) Make a frequency distribution table of the time taken by each of you to solve a problem, (b) find the median from the table.

Mode :

In class VII, we have learned that the number which appears maximum times in a data is the mode of the data. In a data, there may be one or more than one mode. If there is no repetition of a member in a data, data will have no mode. Now we shall discuss how to determine the mode of classified data using formula.

Determining Mode of Classified Data

The formula used to determine the mode of classified data is :

Mode $= L + \frac{f_1}{f_1 + f_2} \times h$, where L is the lower limit of mode-class i.e. the class where the mode lies, $f_1 =$ frequency of mode-class – frequency of the class previous to mode class, $f_2 =$ frequency of mode class – frequency of next class of mode class and $h =$ class interval.

Example 13. Find the mode from the following frequency distribution table.

Class	Frequency
31 – 40	4
41 – 50	6
51 – 60	8
61 – 70	12
71 – 80	9
81 – 90	7
91 – 100	4

Solution

Mode = $L + \frac{f_1}{f_1 + f_2} \times h$

Here, the maximum numbers of repetition of frequency is 12 which lies in the class (61 – 70). Hence, $L = 61$

$f_2 = 12 - 8 = 4$

$f_2 = 12 - 9 = 3$

$h = 10$

$\therefore$ Mode $= 61 + \dfrac{4}{4+3} \times 10 = 61 + \dfrac{4}{7} \times 10$

$= 61 + \dfrac{40}{7} = 61 + 5 \cdot 7 = 66 \cdot 7$

Therefore, the required mode is $66 \cdot 714$

Example 14. Find the mode from the frequency distribution table below :

Class	Frequency
41 – 50	25
51 – 60	20
61 – 70	15
71 – 80	8

Solution : Here, maximum numbers of frequency are 25 which lie in the class (41–50). So it is evident that mode is in this class. We know that

Mode $= L + \dfrac{f_1}{f_1 + f_2} \times h$

Here, $L = 41$ [If the frequency is maximum in the first class, the frequency of previous class is zero]

$f_1 = 25 - 0$

$f_2 = 25 - 20 \quad = 5$

$\therefore$ Mode $= 41 + \dfrac{25}{25+5} \times 10$

$= 41 + \dfrac{25}{30} \times 10 = 51 + 8 \cdot 33$

$= 49 \cdot 33$

Therefore, required mode is $49 \cdot 33$

In classified data, if the first class is mode class the frequency of previous class is considered to be zero.

Example 15. Determine the mode of the following frequency distribution table :

Class	Frequency
10 – 20	4
21 – 30	16
31 – 40	20
41 – 50	25

Solution : The maximum numbers of frequency are 25 which lie in the class (41 –50). So it is obvious that this class is the class of mode. We know that,

Mode $= L + \dfrac{f_1}{f_1 + f_2} \times h$

Here, $L = 41$

$f_1 = 25 - 20 = 5$

$h = 10$

Therefore, mode $= 41 + \frac{5}{25} \times 10$

$= 41 + 2 = 43$

The required mode is 43.

Exercise 17

Put tick (√) mark in the correct answer :

1. Of the following, which one is class interval ?
   (a) The difference between the highest and the lowest data
   (b) The difference between the first and the last data
   (c) The difference between the highest and the lowest number of each class
   (d) The sum of the highest and the lowest numbers of each class.
2. Which one indicates the data included in each class when the data are classified?
   (a) Class limit (b) Mid-point of the class
   (c) Numbers of classes (d) Class frequency
3. If the disorganized data of statistics are arranged according to the value, the data cluster round near any central value. This tendency of data is called
   (a) mode (b) cetral tendency (c) mean (d) median

   In winter, the statistics of temperatures (in celsius) of a region in Bangladesh is 10°, 9°, 8°, 6°, 11°, 12°, 7°, 13°, 14°, 5°. In the context of this statistics, answer the questions from (4 –6).
4. Which is the mode of the above numerical data ?
   (a) 12° (b) 5° (c) 14° (d) no mode
5. Which one is the mean of temperature of the above numerical data ?
   (a) 8° (b) 8.5° (c) 9.5° (d) 9°
6. Which one is the median of the data ?
   (a) 9.5° (b) 9° (c) 8.5° (d) 8°
7. The number of classified data included in the table is n, the lower limit of median class is L, the cumulative data of previous class to median class is F_c, the frequency of median class is f_m and class interval is h. In the light of these information, which one is the formula for determining the median ?
   (a) $L + \left(\frac{n}{2} - F_c\right) \times \frac{h}{f_m}$ (b) $L + \left(\frac{n}{2} - f_m\right) \times \frac{h}{F_m}$
   (c) $L - \left(\frac{n}{2} - F_c\right) \times \frac{h}{f_m}$ (d) $L - \left(\frac{n}{2} - f_n\right) \times \frac{h}{F_m}$

Class Interval	31–40	41–50	51–60	61–80	71–80	81–90	91–100
Frequency	6	12	16	24	12	8	2
Cumulative Frequency	6	18	34	58	70	78	80

8. In how many classes have the data been arranged ?
   (a) 6 (b) 7 (c) 8 (d) 9
9. What is the class interval of the data presented in the table ?
   (a) 5 (b) 9 (c) 10 (d) 15
10. What is the mid value of the 4^{th} class ?
   (a) 71.5 (b) 61.5 (c) 70.5 (d) 75.6
11. Which one is the median class of the data ?
   (a) 41–50 (b) 51–60 (c) 61–70 (d) 71–80
12. What is the cumulative frequency of the previous class to the median class ?
   (a) 18 (b) 34 (c) 58 (d) 70
13. What is the lower limit of median class ?
   (a) 41 (b) 51 (c) 61 (d) 71
14. What is the frequency of median class ?
   (a) 16 (b) 24 (c) 34 (d) 58
15. What is the median of the presented data?
   (a) 63 (b) 63.5 (c) 65 (d) 65.5
16. What is the mode of the presented data ?
   (a) 61.4 (b) 61 (c) 70 (d) 70.4
17. The weights (in kg) of 50 students of class X of a school are :
   45, 50, 55, 51, 56, 57, 56, 60, 58, 60, 61, 60, 62, 60, 63, 64, 60,
   61, 63, 66, 67, 61, 70, 70, 68, 60, 63, 61, 50, 55, 57, 56, 63, 60,
   62, 56, 67, 70, 69, 70, 69, 68, 70, 60, 56, 58, 61, 63, 64.
   (a) Make frequency distribution table considering 5 as a class interval.
   (b) Find the mean from the table in short-cut method.
   (c) Draw frequency polygon of the presented data in frequency distribution table.
18. Frequency distribution table of the marks obtained in mathematics of 50 students of class X are provided. Draw the frequency polygon of the provided data.

Class interval	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Frequency	6	8	10	12	5	7	2

19. The frequency distribution table of a terminal examination in 50 marks of 60 students of a class is as follows :

Marks obtained	1–10	11–20	21–30	31–40	41–50
Frequency	7	10	16	18	9

   Draw an Ogive curve of the data.
20. The frequency distribution table of weights (in kg) are provided below. Determine the median.

Weight (kg)	45	50	55	60	65	70
Frequency	2	6	8	16	12	6

21. The frequency distribution table of weights (in kg) of 60 students of a class are:

Interval	45-49	50-54	55-59	60-64	65-69	70-74
Frequency	4	8	10	20	12	6
Cumulative Frequency	4	12	22	42	54	60

(a) Find the median of the data.
(b) Find the mode of the data.

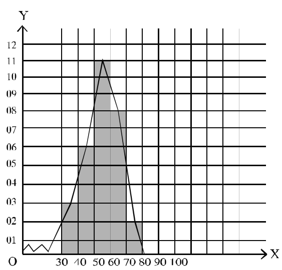
22. In case of data, Mode is-
(i) Measures of central tendency
(ii) Represented value which is mostly occured
(iii) May not unique in all respect
Which is correct on the basis of above information?
a) i and ii   b) i and iii
c) ii and iii   d) i, ii, and iii

23. The following are the marks obtained in Mathematics of fifty students of class IX in a school :

76,	65,	98,	79,	64	68,	56,	73,	83,	57
55,	92,	45,	77,	87	46,	32,	75,	89,	48
97,	88,	65,	73,	93	58,	41,	69,	63,	39
84,	56,	45,	73,	93	62,	67,	69,	65,	53
78,	64,	85,	53,	73	34,	75,	82,	67,	62

(a) What is the type of the given information? What indicate frequency in a class of distribution?
(b) Make frequency table taking appropiate class enterval.
(c) Determine the mean of the given number by shortcut method.

24.



(a) In the above figure, what is class midvalue?
(b) Express by data of information demonstrated in the figure (b).
(c) Find the median of frequency obtained from (b).

Answer

Exercise 1

4. (a) $0\cdot1\dot{6}$ (b) $0\cdot\dot{6}\dot{3}$ (c) $3\cdot\dot{2}$ (d) $3\cdot5\dot{3}$ 5. (a) $\frac{2}{9}$ (b) $\frac{35}{99}$ (c) $\frac{2}{15}$ (d) $3\frac{71}{90}$ (e) $6\frac{769}{3330}$ 6 (a) $2\cdot3\dot{3}\dot{3}$, $5\cdot2\dot{3}\dot{5}$ (b) $7\cdot26\dot{6}$, $4\cdot23\dot{7}$ (c) $5\cdot\dot{7}7777\dot{7}$, $8\cdot\dot{3}4343\dot{4}$, $6\cdot\dot{2}4524\dot{5}$ (d) $12\cdot32\dot{0}\dot{0}$, $2\cdot19\dot{9}\dot{9}$, $4\cdot32\dot{5}\dot{6}$ 7. (a) $0\cdot\dot{5}$ (b) $0\cdot58\dot{9}$ (c) $17\cdot117\dot{9}$ (d) $1\cdot92\dot{6}3\dot{1}$ 8.

(a) $1\cdot3\dot{1}$ (b) $1\cdot6\dot{6}\dot{5}$ (c) $3\cdot13\dot{3}\dot{4}$ (d) $6\cdot11\dot{0}6\dot{2}$ 9. (a) $0\cdot2$ (b) 2 (c) $0\cdot\dot{2}07\dot{4}$ (d) $12\cdot1\dot{8}\dot{5}$ 10. (a) $0\cdot5$ (b) $0\cdot2$ (c) $5\cdot\dot{2}195\dot{1}$ (d) $4\cdot\dot{8}$ 11. (a) $3\cdot4641$, $3\cdot464$ (b) $0\cdot5025$, $0\cdot503$ (c) $1\cdot1595$, $1\cdot160$ (d) $2\cdot2650$, $2\cdot265$

12.(a) Rational (b) Rational (c) Irrational (d) Irrational (e) Irrational (f) Irrational (g) Rational ` (h) Rational 13. (a) 9 (b) 5 (c) 8

Exercise 2.1

1. (a) $\{4, 5\}$ (b) $\{\pm3, \pm4, \pm5, \pm6\}$ (c) $\{6, 12, 18, 36\}$ (d) $\{3, 4\}$

2. (a) $\{x \notin N : x$ is odd number and $1 < \times < 13\}$ (c) $\{x \in N : x,$ in the multiple of 36 $\}$

   (c) $\{x \in N : x,$ is the multiple of 4 and $x \le 40\}$ (d) $\{x \in Z : x^2 \ge 16$ and $x^3 \le 216\}$

3. (a) $\{1\}$ (L) $\{1, 2, 3, 4, a\}$ (b) $\{2\}$ (c) $\{2, 3, 4, a\}$ (d) $\{2\}$

5. $\{\{x, y), (x), (y\}, \phi\}$, $\{\{m, n, l), (m, n), (m, l\}, \{n, l\}, \{m\}, \{n\}, \{l\}, \phi\}$

7. (a) $2, 3$ (L) (a, c) (b) $(1, 5)$

8. (a) $\{(a, b), (a, c)\}, \{(b, a), (c, a)\}$ (b) $\{(4, x), (4, y), (5, x), (5, y)\}$

   (c) $\{3, 3), (5, 3), (7, 3)\}$

9. $\{1, 3, 5, 7, 9, 15, 35, 45\}$ and $\{1, 5\}$ 10. $\{35, 105\}$ 11. 5 persons

Exercise 2·2

4. $\{(3,2),\ (4,2)\}$ 5. $\{(2,4),\ (2,\ 6)\}$ 6. $-7,\ 23,\ \frac{-7}{16}$ 7. 2 8. 1 or 2 or 3 9. $\frac{4}{x}$

11. (a) $\{2\},\ \{1,\ 2,\ 3\}$ (b) $\{-2,-1,\ 0,\ 1,\ 2),(2,-1)\}$

(c) $\left\{\frac{1}{2},\ 1,\ \frac{5}{2}\right\},\ \{0,\ 1,-1,\ 2,\ -2\}$

12. (a) $\{(-1,2),\ (0,1),\ (1,0),\ (2,-1)\},\ \{-1,\ 0,\ 1,2\},\ \{2,\ 1,\ 0,-1\}$

(b) $\{(-1,-2),\ (0,\ 0),\ (1,\ 2)\},\ \{-1,\ 0,\ 1\},\ \{-2,\ 0,2\}$

13. (a) $\sqrt{41}$ (b) 5 (c) 13

Exercise 3·1

1. (a) $4a^2+12ab+9b^2$ (b) $4a^2b^2+12ab^2c+9b^2c^2$ (c) $x^4+\frac{4x^2}{y^2}+\frac{4}{y^4}$

(d) $a^2+2+\frac{1}{a^2}$ (e) $16y^2-40xy+25x^2$ (f) $a^2b^2-2abc+c^2$

(g) $25x^4-10x^2y+y^2$ (h) $x^2+4y^2+16z^2+4xy+16yz+8zx$

(i) $9p^2+16q^2+25r^2+24pq-40qr-30pr$ (j) $9b^2+25c^2+4a^2-30bc+20ca-12ab$

(k) $a^2x^2+b^2y^2+c^2z^2-2abxy+2bcyz-2cazx$

(l) $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$

(m) $4a^2+9x^2+4y^2+25z^2+12ax-8ay-20az-12xy-30xz+20yz$ (n) 10201

(o) 994009 (p) 10140491

2. (a) $16a^2$ (b) $36x^2$ (c) p^2+49r^2-14rp (d) $36n^2-24pn+4p^2$ (e) 100

(f) 4410000 (g) 10 (h) 3104

3. ± 16 4. ± 1 5. $\pm 3m$ 6. 130 8. $\frac{1}{4}$ 11. 19 12. 25 13. 6 14. 138

15. 9 17. $(2a+b+c)^2-(b-a-c)^2$ 18. $(x-1)^2-8^2$ 19. $(x+5)^2-1^2$ 20. (i) 3

Exercise 3·2

1. (a) $8x^3+60x^2+150x+125$ (b) $8x^6+36x^4y^2+54x^2y^4+27y^6$

   (c) $64a^3-240a^2x^2+300ax^4-125x^6$ (d) $343m^6-294m^4n+84m^2n^2-8n^3$

   (e) 65450827 (P) 994011992

   (f) $8a^3-b^3-27c^3-12a^2b-36a^2c+6ab^2+54ac^2-9b^2c-27bc^2+36abc$

   (g) $8x^3+27y^3+z^3+36x^2y+12x^2z+54xy^2+27y^2z+6xz^2+9yz^2+36xyz$

2. (a) $8a^3$ (b) $64x^3$ (c) $8x^3$ (d) 1 (e) $8(b+c)^3$ (f) $64m^3n^3$ (g) $2(x^3+y^3+z^3)$

   (h) $64x^3$

3. 665 4. 54 5. 8 6. 42880 7. 1728 10. (a) 3 (b) 9 11. (a) 133

   (b) 665

12. a^3-3a 13. p^3+3p 13. $46\sqrt{5}$

Exercise 3·3

1.	$(a+b)(a+c)$	2.	$(b+1)(a-1)$
3.	$2(x-y)(x+y+z)$	4.	$b(x-y)(a-c)$
5.	$(3x+4)^2$	6.	$(a^2+5a-1)(a^2-5a-1)$
7.	$(x^2+2xy-y^2)(x^2-2xy-y^2)$	8.	$(ax+by+ay-by)(ax+bx-ay+bx)$
9.	$(2a-3b+2c)(2a-3b-2c)$	10.	$9(x+a)(x-a)(x+2a)(x-2a)$
11.	$(a+y+2)(a-y+4)$	12.	$(4x-5y)(4x+5y-2z)$
13.	$(a+b+c)(b+c-a)(c+a-b)(a+b-c)$	14.	$(x+4)(x+9)$
15.	$(x+2)(x-2)(x^2+5)$	16.	$(a-18)(a-12)$
17.	$(x^3y^3-3)(x^3y^3+2)$	18.	$(a^4-2)(a^4+1)$
19.	$(ab+7)(ab-15)$	20.	$(x+13)(x-15)$
21.	$(x+2)(x-2)(2x+3)(2x-3)$	22.	$(2x-5)(6x-4)$
23.	$y^2(x+1)(9x-14)$	24.	$(x+3)(x-3)(4x^2+9)$

25. $(x+a)(ax+1)$

26. $(a^2+2a-4)(3a^2+6a-10)$

27. $(2z-3x-5)(10x+7z+3)$

28. $-(3a+17b)(9a+7b)$

29. $(x+ay+y)(ax-x+y)$

30. $3x(2x-1)(4x^2+2x+1)$

31. $(a+b)^2(a^4-2a^3b+6a^2b^2-2ab^3+b^4)$

32. $(x+2)(x^2+x+1)$

33. $(a-3)(a^2-3a+3)$

34. $(a-b)(2a^2+5ab+8b^2)$

35. $(2x-3)(4x^2+12x+21)$

36. $\frac{1}{27}(6a+b)(36a^2-6ab+b^2)$

37. $\frac{1}{8}(2a-1)(4a^2+2a+1)$

38. $\left(\frac{a^2}{3}-b^2\right)\left(\frac{a^4}{9}+\frac{a^2b^2}{3}+b^4\right)$

39. $\left(2a-\frac{1}{2a}\right)\left(2a-\frac{1}{2a}+2\right)$

40. $(a+4)(19a^2-13a+7)$

41. $(x+6)(x-10)$

42. $(x^2+7x+4)(x^2+7x-18)$

43. $(x^2-8x+20)(x^2-8x+2)$

Exercise 3·4

1. $(6x-1)(x-1)$

2. $(a+1)(3a^2-3a+5)$

3. $(x+y)(x-3y)(x+2y)$

4. $(x-6)(x+1)$

5. $(2x-3)(x+1)$

6. $(x-3)(3x+2)$

7. $(x-2)(x+1)(x+3)$

8. $(x-1)(x+2)(x+3)$

9. $(a+3)(a^2-3a+12)$

10. $(a-1)(a-1)(a^2+2a+3)$

11. $(a+1)(a-4)(a+2)$

12. $(x-2)(x^2-x+2)$

13. $(a-b)(a^2-6ab+b^2)$

14. $(x-3)(x^2+3x+8)$

15. $(x+y)(x+3y)(x+2y)$

16. $(x-2)(2x+1)(x^2+1)$

17. $(2x-1)(x+1)(x+2)(2x+1)$

18. $x(x-1)(x^2+x+1)(x^2-x+1)$

19. $(4x-1)(x^2-x+1)$

20. $(2x+1)(3x+2)(3x-1)$

Exercise 3·5

1. (c)	2. (d)	3. (b)	4. (b)
5. (d)	6. (c)	7. (d)	8. (d)
9. (a)	10. (c)	11. (d)	12. (b)
13. (a)	14. (b)	15. (c)	16. (b)
17. (a)	18. (b)	19. (c)	20. (d)
21. (2) (b)	21. (3) (d)	22. $\frac{2}{3}(p+r)$ days	23. 5 hours

24. $\frac{xy}{x+y}$ days 25. 95 persons

26. Speed of current is $\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)$ km per hour and speed of boat is $\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}\right)$ km. per hour.

27. The speed of the oar is 8 km/hour and the speed of current is 2 km/hour

28 $\frac{t_1 t_2}{t_2 - t_1}$ minutes 29. 240 liter 30. Tk. 10 31. Tk. 48 32. (a) Tk. 120, (b) Tk. 80, (c) Tk. 60 33. Purchase value Tk. 450 34. 4% 35. Tk. 625 36. Tk. 5%

37. Tk. $522 \cdot 37$ (approx.) 38. Tk. 780 39. Tk. 61

40. VAT is Tk. $\frac{px}{100+x}$; the amount of VAT is Tk. 300.

Exercise 4·1

1. $\frac{10}{7}$	2. $\frac{ab}{3a+2b}$	3. 27	4. $\frac{a^2}{b}$	5. 343	6. 1
7. 4	8. $\frac{1}{9}$	17. $\frac{3}{2}$	18. 3	19. 5	20. $0, 1$

Exercise 4·2

1. (a) 4 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (c) 4 (e) $\frac{5}{6}$

2. (a) 125 (b) 5 (c) 4

4. (a) $\ell og\,2$ (b) $\frac{13}{15}$ (c) 0

Exercise 4·3

1. b 2. d 3. c 4. a 5. c 7. d 8. (1) d (2) c (3) a

9. (a) $6\cdot530\times10^{3}$ (b) $6\cdot0831\times10^{3}$ (c) $2\cdot45\times10^{-4}$ (d) $3\cdot75\times10^{7}$
(e) $1\cdot4\times10^{-7}$

10. (a) 100000 (b) $0\cdot000001$ (c) 25300 (d) $0\cdot009813$ (e) $0\cdot0000312$

11. (a) 3 (b) 1 (c) 0 (d) $\bar{2}$ (e) $\bar{5}$

12. (a) characteristics 1, Mantissa $\cdot43136$ (b) characteristics 1, Mantissa $\cdot80035$
(c) characteristics 0, Mantissa $\cdot14765$ (d) characteristics $\bar{2}$, Mantissa $\cdot65896$
(e) characteristics $\bar{4}$, Mantissa $\cdot82802$

13. (a) $1\cdot66706$ (b) $\bar{1}\cdot64562$ (c) $0\cdot81358$ (d) $\bar{3}\cdot78888$

14. (a) $0\cdot95424$ (b) $1\cdot44710$ (c) $1\cdot62325$

15. a.2 $a^3\cdot5^3$ b. $6\cdot25\times10$ c. characteristics 1, Mantissa $\cdot79588$

Exercise 5·1

1. 1 2. ab 3. -6 4. -1 5. $-\frac{3}{5}$ 6. $-\frac{5}{2}$

7. $\frac{a+b}{2}$ 8. $a+b$

9. $\frac{a+b}{2}$ 10. $\sqrt{3}$ 11. $\{2\}$ 12. $\{4(1+\sqrt{2})\}$

13. $\{-a\}$ 14. ϕ

15. $\left\{-\frac{1}{3}\right\}$ 16. $\left\{\frac{m+n}{2}\right\}$ 17. $\left\{-\frac{7}{2}\right\}$ 18. $\{6\}$ 19. $\left\{(a^2+b^2+c^2)\right\}$

20. 28, 70 21. $\frac{3}{4}$ 22. 72 23. 72 24. 18 25. 9

26. Number of coin of twenty five and fifty paisa are 100 and 20 respectively.
22. 120 km.

Exercise 5·2

1. c 2. b 3. b 4. c 5 d 6. b 7. a 8. (1) d (2) c (3) a

9. $-2,\ \sqrt{3}$ 10. $-\frac{3\sqrt{2}}{2},\ \frac{2\sqrt{3}}{3}$ 11. $-1,\ 6$ 12. ± 7 13. $-6,\ \frac{3}{2}$

14. $1,\ -\frac{3}{20}$

15. $\frac{1}{2},\ 2$ 16. $0,\ \frac{2}{3}$ 17. $\pm\sqrt{ab}$ 18. $0,\ a+b$ 19. $\left\{3,\ -\frac{1}{2}\right\}$

20. $\left\{-\frac{2}{3},\ 2\right\}$

21. $\{-a,\ -b\}$ 22. $\{1,\ -1\}$ 23. $\{1\}$ 24. $\{0,\ 2a\}$ 25. $\left\{\frac{1}{3},\ 1\right\}$ 26. 78 or 87 27. Length 16 metre, breadth 12 metre 28. 9 cm., 12 cm. 29. 27 cm.

30. 21 persons, Tk. 20 31. 70 32. a. $70-9x,\ 9x+7$ b. 34 c. 5 cm. $5\sqrt{2}$ cm.
33. b. 5 cm.. c. $2:5:8$

Exercise 9.1

2. $\cos A=\frac{\sqrt{7}}{4}, \tan A=\frac{3}{\sqrt{7}}, \cot A=\frac{\sqrt{7}}{3}, \sec A=\frac{4}{\sqrt{7}}, \operatorname{cosec} A=\frac{4}{3}$

3. $\sin A=\frac{15}{17}, \cos A=\frac{8}{17}$

4. $\sin\theta=\frac{5}{13}, \cos n\theta=\frac{12}{13}, \tan\theta=\frac{5}{12}$

22. $\frac{1}{2}$ 23. $\frac{3}{4}$ 24. $\frac{4}{3}$ 25. $\frac{a^2\cdot b^2}{a^2+b^2}$

Exercise 9.2

5. $\frac{1}{2}$ 6. $\frac{3}{4\sqrt{2}}$ 7. $\frac{23}{5}$ 8. $\frac{2\sqrt{2}}{3}$ 17. $A=30^\circ, B=30^\circ$ 18. $A=30^\circ$

19. $A=37\frac{1}{2}^\circ, B=7\frac{1}{2}^\circ$ 21. $\theta=90^\circ$ 22. $\theta=60^\circ$ 23. $\theta=60^\circ$ 24. $\theta=45^\circ$ 25. 3

Exercise 10

7. 45.033 metre (app.) 8. 34.641 metre (app.) 9. 12.728 metre (app.)
10. 10 metres
11. 21.651 metre (app.) 12. 141.962 metre(app.) 13. 83.138 metre (app.) and 48 metre
14. 34.298 metre (app.) 15. 44.785 metre (app.) 16. (b) 259.808 metre.

Exercise 11·1

1. $a^2 : b^2$, 2. $\sqrt{\pi} : 2$, 3. 45, 60, 4. 20%, 5. 18 : 25, 6. 13 : 7, 8. (i) $\frac{3}{4}$, (ii) $\frac{2ab}{b^2+1}$, (iii) $x=\pm\sqrt{2ab-b^2}$, (iv) 10, (v) $\frac{b}{2a}\left(c+\frac{1}{c}\right)$, (vi) $\frac{1}{2}$, 2, 22. 3

Exercise 11·2

1. a 2. c 3. c 4. b 5. b 6. 24%, 7. 70%, 8. 70%, 9. a Tk. 40, b Tk.60, c Tk. 120, d Tk. 80, 10. 200, 240, 250, 11. 9 cm. 15cm., 21cm., 12. Tk. 315, Tk. 336, Tk. 360, 13. 140, 14. 81 runs 54 runs, 36 runs, 15. Officer Tk. 24000, Clark Tk. 12000 bearer Tk. 6000 16. 70, 17. 44%, 18. 1% 19. 532 quintal, 20. 8 : 9, 21. 1440 sq.metre, 22. 13 : 12.

Exercise 12.1

1. Consistent, not dependent, single solution 2. Consistent, dependent, innumerable solution 3. inconsistent not dependent, has no solution 4. consistent, dependent, innumerable solution 5. consistent, not dependent,

single solution 6. inconsistent, not dependent, has no solution 7. Consistent, dependent, innumerable solution 8. Consistent, dependent, innumerable solution 9. Consistent, dependent, single solution 10. Consistent, not dependent, single solution .

Exercise 12.2

1. (4, -1) 2. $(\frac{6}{5}, \frac{6}{5})$ 3. (a, b) 4. (4, -1) 5. (1, 2) 6. $\left(\frac{a\,(b-c)}{a\,(b-a)}, \frac{c\,(c-a)}{b\,(b-a)}\right)$
7. $(-\frac{17}{2}, 4)$ 8. (2, 3) 9. (3, 2) 10. $(\frac{5}{2}, -\frac{22}{3})$ 11. (1, 2) 12. (2, -1) 13. (a, b)
14. (2, 4) 15. (4, 5)

Exercise 12.3

1. (2, 2) 2. (2, 3) 3. (-7, 3) 4. (4, 5) 5. (2, 3) 6. (1.5, 1.5) 7. $(1, \frac{1}{2})$ 8. (2, 6)
9. -2 10. 2

Exercise 12.4

1. a 2. c 3. b 4. b 5. b 6. b 7(1) . c 7(2). d 7(3) d 8. $\frac{7}{9}$ 9. $\frac{15}{26}$ 10. 27 11. 37 or 73 12. 30 years 13. length 17 m. breadth 9 metre 14. spread of boat 10 km. per hour, spead of corrent 5 km. per hour 15. starting salary Tk. 4000, yearly increment Tk. 25 16. a. one b. (4, 6) c. sq.unit 17. a. $\frac{x+7}{y} = 2$, $\frac{x}{y-2} = 1$, b. (3, 5), $\frac{3}{5}$

Exercise 13.1

1. -7 Ges -75, 2. 129 Zg, 3. 100 Zg, 4. $p^2 + pq + q^2$, 5. 0, 6. n^2, 7. 360, 8. 320,
9. 42 , 10. 1771, 11. 620, 12. 18, 13. 50, 14. $2+4+6+........$, 15. 110, 16. 0,
17. $-(m+n)$, 20. 50.

Exercise 13.2

5. $\frac{1}{2}$, 2. 3 6. $(3^{14}-1)$, 7. 9th term, 8. $\frac{1}{\sqrt{3}}$, 9. 9th term, 10. $x = 15$, $y = 45$,

11. $x=9,\ y=27,\ z=81$, 12. 86, 13. 1, 14. 55log 2, 15. 650log 2, 16. $n=7$, 17. 0, 18. $n=6$, S = 21, 19 $n=5$, S = 165, 20. $\frac{3069}{512}$, 21. 20, 22. 24.47mm (app.)

Exercise 16·1

1. 20 m., 15 m. 2. 12 m. 3. 12 sq. m. 4. 327·26 sq. m. (app.) 5. 5 m.
6. 30^o 7. 36 or 12 cm. 8. 12 or 16 m. 9 44·44 km. (app.)
10. 24·249 cm. (app.) 254·611 sq. cm. (app.)

Exercise 16·2

1. 9 m. 2. 1056 sq. m. 3. 30 m. and 20 m. 4. 400 m.
5. 6400 6. 16 m. and 10 m. 7. 16·5 m. and 22 m. 8. 35·35 m. (app.)
9 48·66 cm. (app.) 10. 72 cm., 194 sq. cm. 11. 17 cm. and 9 cm.
12. 9·75 sq. cm. (app.) 13. 6·36 sq. m. (app.)

Exercise 16.3

1. 32·97 cm. (app.) 2. 31·513 m. (app.) 3. 20·008 (app.) 4. 128·282 sq. cm. (app.) 5. 7·003 m. (app.) 6. 175·9 m. (app.) 7. 20 times
8. 49·517 m. (app.) 9 $3\sqrt{3}:\pi$

Exercise 16·4

8. 636 sq. m., 20·5 m., 864 cubic metre. 9 14040 sq. m. 10. 12 m., 4 m.
11. 1 cm. 12. 300000 13. 34·641 sq. cm. 14. 534·071 sq. cm. (app.)
942·48 cubic cm. (app.) 15. 5·305 sq. cm. 3 cm.. 16. 6111·8 sq. cm.
17. 147·027 kg. (app.)

Exercise 17

1. (c) 2. (b) 3. (b) 4. (d) 5. (c) 6. (a) 7. (a) 8. (b) 9(c)
10. (c) 11. (c) 12. (c) 13. (c) 14. (b) 15. (b) 16. (a) 20. Median
60 21. (a)62 kg , (b)62.8 kg

2013

Academic Year

9-10 Math

সমৃদ্ধ বাংলাদেশ গড়ে তোলার জন্য যোগ্যতা অর্জন কর

– মাননীয় প্রধানমন্ত্রী শেখ হাসিনা

জ্ঞান মানুষের অন্তরকে আলোকিত করে

জাতীয় শিক্ষাক্রম ও পাঠ্যপুস্তক বোর্ড
ঢাকা বাংলাদেশ

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